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T-TRANSMUTED X FAMILY OF DISTRIBUTIONS

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1. INTRODUCTION

Lifetime distributions are used to explain the life of a system, a device, and in general, time-to-event data. These distributions are frequently used in the fields like reliability, biology, engineering, insurance, etc. The distributions such as exponential, gamma, Weibull have been frequently used in statistical literature to analyze lifetime data. Nadarajah *et al.* (2013) introduced a family of lifetime models by adding a parameter to the Marshall-Olkin family of distributions. Jayakumar and Babu (2015) introduced a class of distributions containing Marshall-Olkin extended Weibull distribution and studied the role of this distribution in the study of minification process. Babu (2016) introduced Weibull-truncated negative binomial ($WTNB$) distribution and studied the application of this distribution in medical sciences.

According to the quadratic rank transmutation map ($QRTM$) approach by Shaw and Buckley (2007), the cumulative distribution function (cdf) satisfy the relationship

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2; \ |\lambda| \leq 1, \tag{1}$$

where $G(x)$ is the cdf of the base distribution. When $\lambda = 0$, we get the cdf of the base random variable. Differentiating (1) yields

$$f(x) = g(x)[1+\lambda - 2\lambda G(x)]; \ |\lambda| \leq 1, \tag{2}$$

where $f(x)$ and $g(x)$ are the probability density functions corresponding to $F(x)$ and $G(x)$ respectively. The survival function of (1) is given by

$$\bar{F}(x) = 1 - F(x) = 1 - G(x)[1+\lambda\bar{G}(x)]; \ |\lambda| \leq 1, \tag{3}$$

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where $\bar{G}(x) = 1 - G(x)$.

Recently various research papers have been appeared in the literature on transmuted generalizations of distributions. Some of them are: transmuted extreme value distribution by Aryal and Tsokos (2009), transmuted Weibull distribution by Aryal and Tsokos (2011), transmuted modified Weibull distribution by Khan and King (2013a), transmuted generalized inverse Weibull distribution by Khan and King (2013b), transmuted log-logistic distribution by Aryal (2013), transmuted additive Weibull distribution by Elbatal and Aryal (2013) and transmuted Weibull Lomax by Afify *et al.* (2015).

Alzaatreh *et al.* (2013b) developed a method to generate a family of continuous distributions called $T-X$ family of distributions. The $T-X$ family is a method for generating generalized distributions of X using T. The random variable X is known as "the transformer" and the random variable T is known as "the transformed". The resulting family has a connection with the hazard functions where each generated distributions is considered as weighted hazard function of the random variable X. Several known continuous distributions are found to be special cases of this family. This family is defined as follows:

Let $r(t)$ be the probability density function (pdf) of a random variable $T \epsilon [a,b]$, for $-\infty \leq a < b \leq \infty$ and $W(F(x))$ be a function of the cdf $F(x)$ of any random variable X which satisfies the following conditions:

$$\left.\begin{array}{l} W(F(x)) \,\epsilon\, [a,b], \\ W(F(x)) \text{ is absolutely continuous and monotonically non-decreasing,} \\ W(F(x)) \to a \text{ as } x \to -\infty \text{ and } \quad W(F(x)) \to b \text{ as } x \to \infty. \end{array}\right\} \tag{4}$$

The cdf of $T-X$ family of distributions is defined as

$$J(x) = \int_a^{W(F(x))} r(t)dt. \tag{5}$$

where $W(F(x))$ satisfies the conditions in (4). Here the cdf $J(x)$ can be written as $J(x) = R\{W(F(x))\}$, where $R\{.\}$ is the cdf of the random variable T. The pdf corresponding to (5) is

$$j(x) = \left\{\frac{d}{dx} W(F(x))\right\} r\{W(F(x))\}. \tag{6}$$

Aljarrah *et al.* (2014) introduced a wider class of $W(.)$ functions defined in (4) as $W : (0,1) \to (a,b)$, where $-\infty \leq a < b \leq \infty$, is right continuous and non decreasing function, such that, $\lim_{y\to 0+} W(y) = a$ and $\lim_{y\to 1-} W(y) = b$. Now $J(x), -\infty < x < \infty$, is a distribution function satisfies the following conditions:

i) $J(x)$ is non-decreasing,

ii) $J(x)$ is right continuous and

iii) $J(x) \to 0$ as $x \to -\infty$ and $J(x) \to 1$ as $x \to \infty$.

Let $W(F(x)) = -\ln(1-F(x))$, and the random variable T be defined on $(0,\infty)$. Then the cdf of the $T-X$ family of distributions becomes

$$J(x) = \int_0^{-\ln(1-F(x))} dR(t). \tag{7}$$

Several research papers are appeared in the literature based on the $T-X$ family introduced by Alzaatreh *et al.* (2013b). Some of them are the Weibull-Pareto distribution by Alzaatreh *et al.* (2013a), gamma-half normal distribution by Alzaatreh and Knight (2013), Weibull-X family by Alzaatreh and Ghosh (2015), beta Marshall-Olkin family by Alizadeh *et al.* (2015), generalized transmuted family by Alizadeh *et al.* (2017), etc.

In this paper, we introduce a combined family of $T-X$ and transmuted distributions. The results of this paper is organized as follows: In Section 2, we introduce a new family of distributions called "T-transmuted X family" and study its properties. Some members of T-transmuted X family are identified in Section 3. Properties of one of the members of T-transmuted X family called, Exponential-transmuted Exponential (ETE) distribution are studied in Section 4. Two real data sets are analyzed in Section 5, to show the flexibility of ETE distribution to model life time data. Finally, conclusions are given in Section 6.

2. T-TRANSMUTED X FAMILY OF DISTRIBUTIONS

In the composite function $W(F(x))$ defined in (4), if we take $F(x)$ as the transmuted family defined (1), we get several family of distributions. As a special case, we take $W(F(x)) = -\ln[\bar{F}(x)]$, the cumulative hazard function of $F(x)$, where $F(x)$ is a transmuted family of distributions given in (1). That is

$$W(F(x)) = -\ln\left[1 - G(x)[1+\lambda\bar{G}(x)]\right].$$

Then from (7), the cdf of the new family is

$$J(x) = \int_0^{-\ln\left[1-G(x)[1+\lambda\bar{G}(x)]\right]} dR(t) = R\left\{-\ln\left[1-G(x)[1+\lambda\bar{G}(x)]\right]\right\}, \tag{8}$$

where $R(t)$ is the cdf of the random variable T with pdf $r(t)$. We call $J(x)$ as the "T-transmuted X family" of distributions.
The pdf of (8) is

$$j(x) = \frac{d}{dx}(J(x)) = \frac{g(x)[1+\lambda-2\lambda G(x)]}{1-G(x)[1+\lambda\bar{G}(x)]} \quad r\left\{-\ln\left[1-G(x)[1+\lambda\bar{G}(x)]\right]\right\}. \tag{9}$$

The hazard rate function (hrf) is given by

$$h(x) = \frac{j(x)}{1-J(x)} = \frac{g(x)[1+\lambda-2\lambda G(x)]}{1-G(x)[1+\lambda\bar{G}(x)]} \frac{r\left\{-\ln[1-G(x)(1+\lambda\bar{G}(x))]\right\}}{1-R\left\{-\ln[1-G(x)(1+\lambda\bar{G}(x))]\right\}}. \tag{10}$$

The shapes of the density and hazard rate functions of T-transmuted X family can be described analytically. The critical points of the density function are the roots of the equation

$$\frac{\partial \ \ln(j(x))}{\partial x} = \frac{g'(x)}{g(x)} - \frac{2\lambda g(x)}{1+\lambda-2\lambda G(x)} - \frac{g(x)[1+\lambda-2\lambda G(x)]}{1-G(x)[1+\lambda\bar{G}(x)]} \left[\frac{1}{r\left\{-\ln[1-G(x)(1+\lambda\bar{G}(x))]\right\}}+1\right] = 0. \tag{11}$$

Here (11) may have more than one root. If $x = x_0$ is a root of (11), then it corresponds to a local maximum if $\frac{\partial^2 \ln(j(x))}{\partial x^2} < 0$, a local minimum if $\frac{\partial^2 \ln(j(x))}{\partial x^2} > 0$, and a point of inflection if $\frac{\partial^2 \ln(j(x))}{\partial x^2} = 0$.

Similarly, the critical points of $h(x)$ are the roots of the equation

$$\frac{\partial \ \ln(h(x))}{\partial x} = \frac{g'(x)}{g(x)} - \frac{2\lambda g(x)}{1+\lambda-2\lambda G(x)} - \frac{g(x)[1+\lambda-2\lambda G(x)]}{1-G(x)[1+\lambda\bar{G}(x)]}\left[\frac{1}{r\left\{-\ln[1-G(x)(1+\lambda\bar{G}(x))]\right\}} + \frac{r\left\{-\ln[1-G(x)(1+\lambda\bar{G}(x))]\right\}}{1-R\left\{-\ln[1-G(x)(1+\lambda\bar{G}(x))]\right\}}+1\right] = 0. \tag{12}$$

There may be more than one root to (12). If $x = x_0$ is a root of (12), then it corresponds to a local maximum if $\frac{\partial^2 \ln(h(x))}{\partial x^2} < 0$, a local minimum if $\frac{\partial^2 \ln(h(x))}{\partial x^2} > 0$, and a point of inflection if $\frac{\partial^2 \ln(h(x))}{\partial x^2} = 0$.

Several family of distributions can be derived from T-transmuted X family for different choices of $r(t)$. Some of them are given below:

i) When $r(t)$ follows exponential distribution with parameter θ. We have $r(t) = \theta e^{-\theta t}$; $t > 0, \theta > 0$. Then from (9)

$$j(x) = \theta g(x)\big[1+\lambda-2\lambda G(x)\big]\big[1-G(x)(1+\lambda\bar{G}(x))\big]^{\theta-1}. \tag{13}$$

ii) When $r(t)$ is exponentiated exponential distribution with parameters θ and α. We have $r(t) = \frac{\alpha\theta(1-e^{-\theta t})^{\alpha-1}}{e^{\theta t}}$; $t > 0, \theta > 0, \alpha > 0$. Then

$$j(x) = \frac{\theta g(x)\big[1+\lambda-2\lambda G(x)\big]\big[1-(1-G(x)(1+\lambda\bar{G}(x)))^{\theta}\big]^{\alpha-1}}{[1-G(x)(1+\lambda\bar{G}(x))]^{1-\theta}}. \tag{14}$$

iii) When $r(t)$ is beta-exponential with parameters θ, α and β.
We have $r(t) = \frac{\theta e^{-\theta\beta t}(1-e^{-\theta t})^{\alpha-1}}{B(\alpha,\beta)}$; $t > 0, \theta > 0, \alpha > 0, \beta > 0$.

Then

$$j(x)=\frac{\theta g(x)[1+\lambda-2\lambda G(x)]}{B(\alpha,\beta)}\frac{\left[1-(1-G(x)(1+\lambda\bar{G}(x)))^{\theta}\right]^{\alpha-1}}{[1-G(x)(1+\lambda\bar{G}(x))]^{1-\theta\beta}}. \tag{15}$$

iv) When $r(t)$ is gamma distribution with parameters α and β.
We have $r(t)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}t^{\alpha-1}e^{-\frac{t}{\beta}};\ t>0,\alpha>0,\beta>0$.
Then

$$j(x)=\frac{g(x)}{\Gamma(\alpha)\beta^{\alpha}}[1+\lambda-2\lambda G(x)][1-G(x)(1+\lambda\bar{G}(x))]^{\frac{1}{\beta}-1}. \tag{16}$$

v) When $r(t)$ is half normal distribution with parameter σ.
We have $r(t)=\frac{1}{\sigma}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}e^{-\frac{t^2}{2\sigma^2}};\ t>0,\sigma>0$.
Then

$$j(x)=\frac{1}{\sigma}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{g(x)[1+\lambda-2\lambda G(x)]}{1-G(x)(1+\lambda\bar{G}(x)}e^{-\frac{1}{2\sigma^2}[\ln(1-G(x)(1+\lambda\bar{G}(x)))]^2}. \tag{17}$$

vi) When $r(t)$ is Levy distribution with parameter α.
We have $r(t)=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\frac{e^{-\frac{\alpha}{2t}}}{t^{\frac{3}{2}}};\ t>0,\alpha>0$.
Then

$$j(x)=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\frac{g(x)[1+\lambda-2\lambda G(x)]e^{\frac{\alpha}{2\ \ln(1-G(x)(1+\lambda\bar{G}(x)))}}}{[1-G(x)(1+\lambda\bar{G}(x))][-\ln(1-G(x)(1+\lambda\bar{G}(x)))^{\frac{3}{2}}]}. \tag{18}$$

vii) When $r(t)$ is log-logistic distribution with parameters α and β.
We have $r(t)=\frac{\beta(\frac{t}{\alpha})^{\beta-1}}{\alpha\left[1+(\frac{t}{\alpha})^{\beta}\right]^2};t>0,\alpha>0,\beta>0$.
Then

$$j(x)=\frac{\beta\alpha^{\beta}g(x)[1+\lambda-2\lambda G(x)]}{1-G(x)(1+\lambda\bar{G}(x))}\frac{[-\ln[1-G(x)(1+\lambda\bar{G}(x))]]^{\beta-1}}{\left[\alpha^{\beta}+\left[-\ln[1-G(x)(1+\lambda\bar{G}(x))]\right]^{\beta}\right]^2}. \tag{19}$$

viii) When $r(t)$ is Rayleigh distribution with parameter σ.
We have $r(t)=\frac{t}{\sigma^2}e^{-\frac{t^2}{2\sigma^2}};\ t>0,\sigma>0$.
Then

$$j(x)=\frac{-\ln[1-G(x)(1+\lambda\bar{G})]g(x)[1+\lambda-2\lambda G(x)]}{\sigma^2[1-G(x)(1+\lambda\bar{G}(x))]e^{\frac{1}{2\sigma^2}[\ln(1-G(x)(1+\lambda\bar{G}))]^2}}. \tag{20}$$

ix) When $r(t)$ is Type-2 Gumbel distribution with parameters α and β.
We have $r(t)=\alpha\beta t^{-\alpha-1}e^{-\beta t^{-\alpha}};\ t>0,\alpha>0,\beta>0$.

Then

$$j(x)=\alpha\beta\frac{g(x)[1+\lambda-2\lambda G(x)]}{1-G(x)(1+\lambda\bar{G}(x))}\frac{e^{-\beta[-\ln[1-G(x)(1+\lambda\bar{G}(x))]]^{-\alpha}}}{\left[-\ln[1-G(x)(1+\lambda\bar{G}(x))]\right]^{\alpha+1}}. \qquad (21)$$

x) When $r(t)$ is Lomax distribution with parameters θ and $\varkappa$.
We have $r(t)=\frac{\theta\varkappa}{(1+\theta t)^{\varkappa+1}};\ t>0,\theta>0,\varkappa>0.$
Then

$$j(x)=\frac{\theta\varkappa\ g(x)[1+\lambda-2\lambda G(x)]}{\left[1-G(x)(1+\lambda\bar{G}(x))\right]\left[1-\theta\ \ln[1-G(x)(1+\lambda\bar{G}(x))]\right]^{\varkappa+1}}. \qquad (22)$$

xi) When $r(t)$ is Weibull distribution with parameters θ and c.
We have $r(t)=\frac{c}{\theta}(\frac{t}{\theta})^{c-1}e^{-(\frac{t}{\theta})^{c}};\ t>0,\theta>0,c>0.$
Then

$$j(x)=\frac{c}{\theta^{c}}\frac{g(x)[1+\lambda-2\lambda G(x)]}{[1-G(x)(1+\lambda\bar{G}(x))]}\frac{\left[-\ln[1-G(x)(1+\lambda\bar{G}(x))]\right]^{c-1}}{e^{\left(\frac{-\ln[1-G(x)(1+\lambda\bar{G}(x))]}{\theta}\right)^{c}}}. \qquad (23)$$

3. SOME MEMBERS OF T-TRANSMUTED X FAMILY OF DISTRIBUTIONS AND THEIR PROPERTIES

In this section we discuss some members of the T-transmuted X family. Here we consider the case where T follows exponential distribution with parameter $\theta>0$ and the cdf and pdf are respectively, $R(t)=1-e^{-\theta t}$ and $r(t)=\theta e^{-\theta t};\ t>0,\theta>0$.

3.1. Exponential-transmuted exponential (ETE) *distribution*

Let $W(F(x))=-\ln\left(1-\bar{F}(x)\right)=-\ln\left[1-G(x)(1+\lambda\bar{G}(x))\right]$, where the base distribution is exponential with cdf, $G(x)=1-e^{-\beta x};\ x>0,\beta>0.$
Then the cdf of the corresponding family is given by

$$J(x)=1-e^{-\theta\beta x}(1-\lambda+\lambda e^{-\beta x})^{\theta};\ x>0,\theta>0,\beta>0,|\lambda|\leq 1. \qquad (24)$$

We call this new family of distributions as exponential-transmuted exponential (ETE) distribution with parameters θ,β and λ.
The pdf of this distribution is

$$j(x)=\theta\beta e^{-\theta\beta x}\frac{\left(1-\lambda+2\lambda e^{-\beta x}\right)}{\left(1-\lambda+\lambda e^{-\beta x}\right)^{1-\theta}};\ x>0,\theta>0,\beta>0,|\lambda|\leq 1. \qquad (25)$$

When $\lambda=0$, ETE distribution becomes the well known exponential distribution. The shapes of the pdf of ETE distribution for various parameter values are shown in Figure 1.

3.2. Exponential-transmuted uniform (ETU) distribution

We consider the base distribution as uniform distribution with cdf and pdf are given by, $G(x)=\frac{x}{\alpha}$ and $g(x)=\frac{1}{\alpha};\ 0<x<\alpha$. Then the cdf and pdf of ETU distribution are given by

$$J(x)=1-\Big[1-\frac{x}{\alpha}[1+\lambda(1-\frac{x}{\alpha})]\Big]^{\theta} \tag{26}$$

and

$$j(x)=\frac{\theta}{\alpha}\Big[1-\frac{x}{\alpha}[1+\lambda(1-\frac{x}{\alpha})]\Big]^{\theta-1}\Big(1+\lambda\big(1-\frac{2x}{\alpha}\big)\Big), \tag{27}$$

where, $\alpha>0,\theta>0,|\lambda|\leq 1$ and $0<x<\alpha$.
Shape of pdf of ETU distribution for various parameters are shown in Figure 2.

3.3. Exponential-transmuted Fréchet (ETF) distribution

Here we consider the base distribution as Fréchet distribution with cdf and pdf are given by, $G(x)=e^{-(\frac{\beta}{x})^{\alpha}}$ and $g(x)=\alpha\beta^{\alpha}x^{-(\alpha+1)}e^{-(\frac{\beta}{x})^{\alpha}};\ x>0,\alpha>0,\beta>0$. Then $W(F(x))=-\ln\big(1-e^{-(\frac{\beta}{x})^{\alpha}}\big[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})\big]\big)$.
Now the cdf and pdf of ETF distribution are given by

$$\begin{aligned} J(x) &= R\Big[-\ln\big(1-e^{-(\frac{\beta}{x})^{\alpha}}\big[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})\big]\big)\Big] \\ &= 1-\Big[1-e^{-(\frac{\beta}{x})^{\alpha}}\big[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})\big]\Big]^{\theta} \end{aligned} \tag{28}$$

and

$$j(x)=\theta\alpha\beta^{\alpha}x^{-(\alpha+1)}e^{-(\frac{\beta}{x})^{\alpha}}\frac{(1+\lambda-2\lambda e^{-(\frac{\beta}{x})^{\alpha}})}{(1-e^{-(\frac{\beta}{x})^{\alpha}}[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})])^{1-\theta}}, \tag{29}$$

where, $\alpha>0,\beta>0,\theta>0,|\lambda|\leq 1$ and $x>0$. Shape of pdf of ETF distribution for various parameters are shown in Figure 3.

3.4. Exponential-transmuted Rayleigh (ETR) distribution

We consider the base distribution as Rayleigh distribution with cdf and pdf are given by $G(x)=1-e^{-\frac{x^2}{2\sigma^2}}$ and $g(x)=\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$. Then the cdf and pdf of ETR distribution are given by

$$J(x)=1-e^{-\frac{\theta x^2}{2\sigma^2}}\big[1-\lambda+\lambda e^{-\frac{x^2}{2\sigma^2}}\big]^{\theta} \tag{30}$$

and

$$j(x) = \frac{\theta x e^{-\frac{\theta x^2}{2\sigma^2}}}{\sigma^2} \frac{\left[1-\lambda+2\lambda e^{-\frac{x^2}{2\sigma^2}}\right]}{\left[1-\lambda+\lambda e^{-\frac{x^2}{2\sigma^2}}\right]^{1-\theta}}, \tag{31}$$

where, $\sigma > 0, \theta > 0, |\lambda| \leq 1$ and $x > 0$. Shape of pdf of ETR distribution for various parameters are shown in Figure 4.

3.5. Exponential-transmuted Weibull (ETW) distribution

Here we consider the base distribution as Weibull distribution with cdf and pdf are given by $G(x) = 1 - e^{-\beta x^\alpha}$ and $g(x) = \alpha\beta x^{\alpha-1} e^{-\beta x^\alpha}$. Then the cdf and pdf of ETW distribution are given by

$$J(x) = 1 - e^{-\theta\beta x^\alpha}\left[1-\lambda+\lambda e^{-\beta x^\alpha}\right]^\theta \tag{32}$$

and

$$j(x) = \theta\alpha\beta x^{\alpha-1} e^{-\theta\beta x^\alpha} \frac{(1-\lambda+2\lambda e^{-\beta x^\alpha})}{(1-\lambda+\lambda e^{-\beta x^\alpha})^{1-\theta}}, \tag{33}$$

where, $\alpha > 0, \beta > 0, \theta > 0, |\lambda| \leq 1$ and $x > 0$. Shape of pdf of ETW distribution for various parameters are shown in Figure 5. In the next section, we study some properties of one of the members of T-transmuted X family, namely ETE distribution.

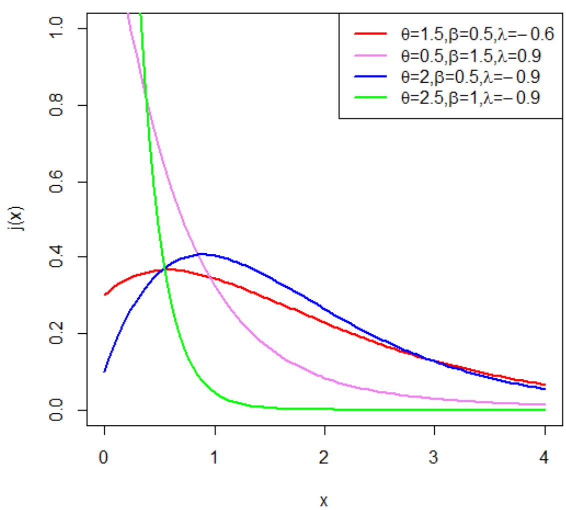

Figure 1 – Pdf of ETE distribution for various choices of α, θ and λ.

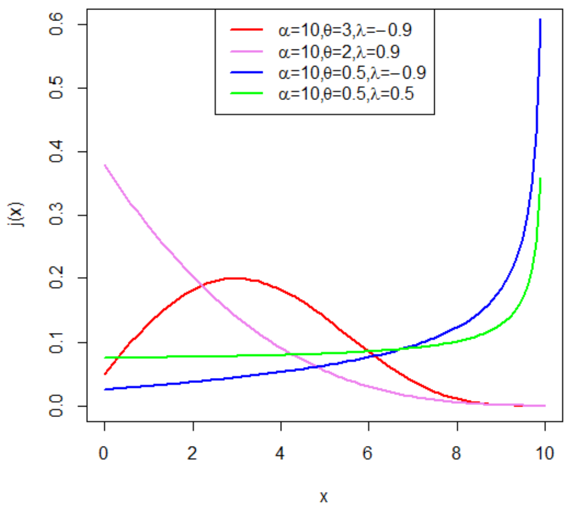

Figure 2 – Pdf of ETU distribution for various choices of α, θ and λ.

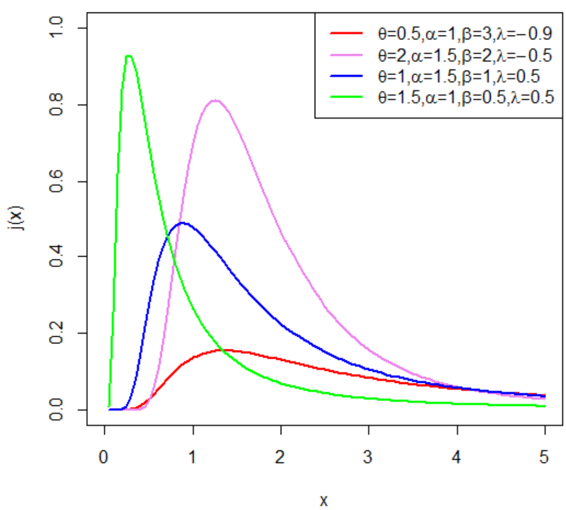

Figure 3 – Pdf of ETF distribution for various choices of θ, α, β and λ.

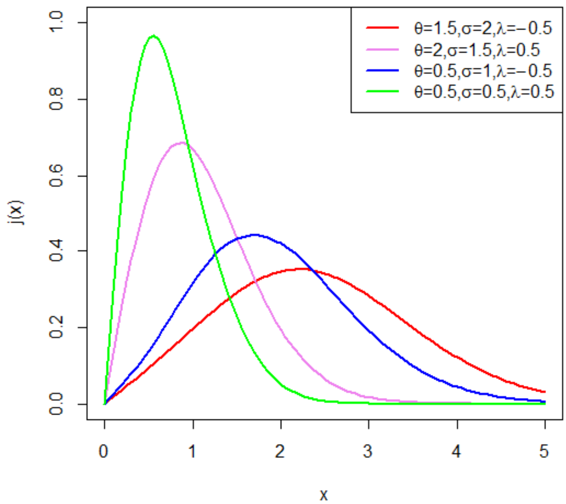

Figure 4 – Pdf of ETR distribution for various choices of θ, σ and λ.

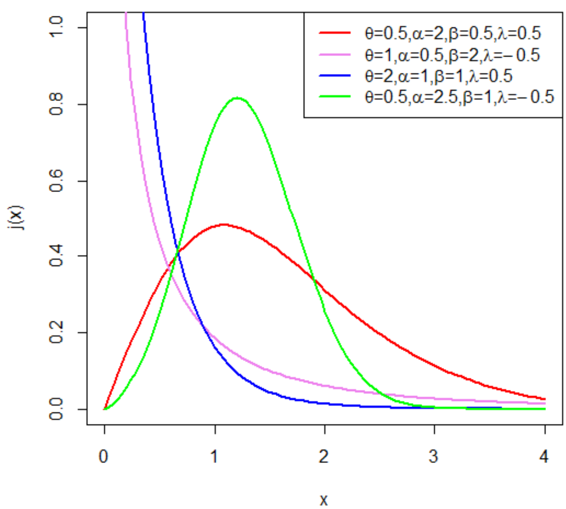

Figure 5 – Pdf of ETW distribution for various choices of θ, α, β and λ.

4. PROPERTIES OF ETE DISTRIBUTION

Using the binomial expansion, the cdf of ETE distribution given in (24) can be expressed as

$$\begin{aligned}
J(x) &= 1-e^{-\theta\beta x}\left(1-\lambda(1-e^{-\beta x})\right)^{\theta} \\
&= 1-e^{-\theta\beta x}\sum_{i=0}^{\infty}(-1)^{i}\binom{\theta}{i}\lambda^{i}(1-e^{-\beta x})^{i} \\
&= 1-e^{-\theta\beta x}\sum_{i=0}^{\infty}(-1)^{i}\binom{\theta}{i}\lambda^{i}\sum_{k=0}^{\infty}(-1)^{k}\binom{i}{k}e^{-k\beta x} \\
&= 1-e^{-\theta\beta x}\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}(-1)^{i+k}\lambda^{i}\binom{\theta}{i}\binom{i}{k}e^{-k\beta x} \\
&= 1-\sum_{k=0}^{\infty}S_k(\theta,\lambda)e^{-(k+\theta)\beta x},
\end{aligned}$$

where $S_k(\theta,\lambda)=\sum_{i=k}^{\infty}(-1)^{i+k}\binom{\theta}{i}\binom{i}{k}\lambda^{i}$.

That is

$$J(x)=1-\sum_{k=0}^{\infty}S_k(\theta,\lambda)e^{-(k+\theta)\beta x}. \tag{34}$$

Then the pdf can be expressed as

$$j(x)=\sum_{k=0}^{\infty}S_k(\theta,\lambda)(k+\theta)\beta e^{-(k+\theta)\beta x}. \tag{35}$$

4.1. *Shapes of the density function*

The shapes of the density function can be described analytically. The critical points of ETE density function are the roots of the equation:

$$\frac{\partial\ \ln(j(x))}{\partial x}=0.$$

That is

$$\frac{\partial\ \ln(j(x))}{\partial x}=-\theta\beta-\frac{\lambda\beta(\theta-1)e^{-\beta x}}{1-\lambda+\lambda e^{-\beta x}}-\frac{2\lambda\beta e^{-\beta x}}{1-\lambda+2\lambda e^{-\beta x}}=0. \tag{36}$$

This implies

$$u^2[4\theta\lambda^2]+u[\lambda(\lambda-1)(4\theta+1)]+\theta(1-\lambda)^2=0, \tag{37}$$

where $u = e^{-\beta x}$. Here (37) is a quadratic equation of u and since, $0 < u < 1$, the possible root of (37) is

$$u = \frac{1-\lambda}{\lambda}\Big[\frac{(8\theta+1)^{\frac{1}{2}} - (4\theta+1)}{8\theta}\Big].$$

Therefore, the solution of (36) is

$$x_0 = \frac{-\ln(u)}{\beta}.$$

Since, $\theta > 0$ and $0 < u < 1$, the root x_0 exist only if $-1 < \lambda < \frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < 0$. Thus the shape of the density function of ETE distribution is unimodal for $x > 0, \theta > 0, \beta > 0$ and $-1 < \lambda < \frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < 0$. Also note that

$$\frac{\partial^2 \ln(j(x))}{\partial x^2} = \lambda\beta^2(1-\lambda)e^{-\beta x}\Big[\frac{\theta-1}{(1-\lambda+\lambda e^{-\beta x})^2} + \frac{2}{(1-\lambda+2\lambda e^{-\beta x})^2}\Big]. \tag{38}$$

Since, $\lambda < 0, \theta > 0, \beta > 0$ and $0 < e^{-\beta x} < 1$ equation (38) is always negative. That is

$$\frac{\partial^2 \ln(j(x))}{\partial x^2} < 0.$$

The third derivative $\frac{\partial^3 \ln(j(x))}{\partial x^3}$ is also exist. The mode of ETE distribution is given by

$$x_0 = \frac{-1}{\beta}\ \ln\Big[\frac{1-\lambda}{\lambda}\Big(\frac{(8\theta+1)^{\frac{1}{2}} - (4\theta+1)}{8\theta}\Big)\Big], \tag{39}$$

where $\theta > 0$ and $-1 < \lambda < \frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < 0$. Thus, the shape of the pdf of ETE is decreasing for $\frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < \lambda < 1$ and is unimodal for $-1 < \lambda < \frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < 0$.

4.2. *Hazard rate function*

The hazard rate function of ETE distribution is given by

$$h(x) = \frac{j(x)}{1-J(x)} = \theta\beta\frac{1-\lambda+2\lambda e^{-\beta x}}{1-\lambda+\lambda e^{-\beta x}};\ x > 0, \theta > 0, \beta > 0, |\lambda| \leq 1. \tag{40}$$

Here note that, $\lim_{x\to 0} h(x) = \theta\beta(\lambda+1)$, and $\lim_{x\to\infty} h(x) = \theta\beta$.
We have the following cases:

Case i. When $-1 \leq \lambda < 0$, $h(x)$ is an increasing function increases from $(1+\lambda)\theta\beta$ to $\theta\beta$.
Case ii. When $\lambda = 0$, $h(x) = \theta\beta$, a constant function.
Case iii. When $0 < \lambda < 1$, $h(x)$ is a decreasing function decreases from $(1+\lambda)\theta\beta$ to $\theta\beta$.
Case iv. When $\lambda = 1$, $h(x) = 2\theta\beta$, a constant function.
The shapes of the hazard rate function for various parameter values are presented in Figure 6.

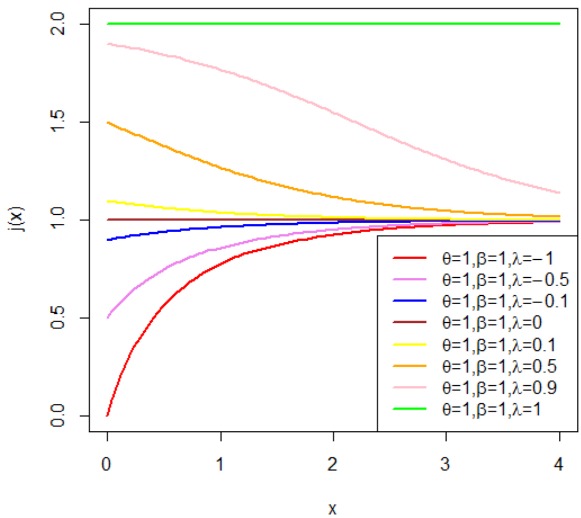

Figure 6 – Hazard rate function of ETE distribution for various parameter values.

4.3. Quantile function

The p^{th}quantile x_p of ETE distribution is the real solution of the equation

$$J(x_p) = p.$$

That is

$$1 - e^{-\theta\beta x_p}(1-\lambda+\lambda e^{-\beta x_p})^{\theta} = p$$

$$\Rightarrow e^{-\beta x_p}(1-\lambda+\lambda e^{-\beta x_p}) = (1-p)^{\frac{1}{\theta}}. \tag{41}$$

Let $u = e^{-\beta x_p}$, then $x_p = -\frac{\ln(u)}{\beta}$.
Then from (41)

$$\lambda u^2 + (1-\lambda)u - (1-p)^{\frac{1}{\theta}} = 0$$

$$\Rightarrow u = \frac{-(1-\lambda) \pm \sqrt{(1-\lambda)^2 + 4\lambda(1-p)^{\frac{1}{\theta}}}}{2\lambda}.$$

Since $0 < u < 1$, the possible root of u is

$$u = \frac{-(1-\lambda) + \sqrt{(1-\lambda)^2 + 4\lambda(1-p)^{\frac{1}{\theta}}}}{2\lambda}.$$

Therefore

$$x_p = -\frac{1}{\beta} \ln\left\{\left[\frac{1}{\lambda}(1-p)^{\frac{1}{\theta}} + \frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}} - \frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right\}. \quad (42)$$

In particular, the median is given by

$$\text{Median} = x_{0.5} = -\frac{1}{\beta} \ln\left\{\left[\frac{1}{\lambda}\left(\frac{1}{2}\right)^{\frac{1}{\theta}} + \frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}} - \frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right\}. \quad (43)$$

4.4. Moments

Here we derive the expression for raw moments of ETE distribution as

$$\begin{aligned} \mu_r^{'} = E(X^r) &= \int_0^\infty x^r \sum_{k=0}^{\infty} S_k(\theta,\lambda)(k+\theta)\beta e^{-(k+\theta)\beta x} dx \\ &= \sum_{k=0}^{\infty} S_k(\theta,\lambda)(k+\theta)\beta \int_0^\infty x^r e^{-(k+\theta)\beta x} dx \\ &= \sum_{k=0}^{\infty} S_k(\theta,\lambda)\frac{\Gamma(r+1)}{[(k+\theta)\beta]^r}. \end{aligned} \quad (44)$$

The first four raw moments are
$\mu_1^{'} = \sum_{k=0}^{\infty} S_k(\theta,\lambda)\frac{1}{[(k+\theta)\beta]}$, $\mu_2^{'} = \sum_{k=0}^{\infty} S_k(\theta,\lambda)\frac{2}{[(k+\theta)\beta]^2}$,
$\mu_3^{'} = \sum_{k=0}^{\infty} S_k(\theta,\lambda)\frac{6}{[(k+\theta)\beta]^3}$, and $\mu_4^{'} = \sum_{k=0}^{\infty} S_k(\theta,\lambda)\frac{24}{[(k+\theta)\beta]^4}$, respectively.
Then, skewness $= \frac{\mu_3^{'2}}{\mu_2^{'3}}$ and kurtosis $= \frac{\mu_4^{'}}{\mu_2^{'2}}$.
Since the pdf of ETE distribution is decreasing for $\frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < \lambda < 1$ it may be skewed to the right. Also for $-1 < \lambda < \frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < 0$ the pdf is unimodal and the mode value is always less than the mean value, it shows the right skewness.

Table 1 gives the raw moments, central moments, mode, skewness and kurtosis of ETE distribution for different choices of parameter values. In all the cases the distribution shows a positively skewed behavior.

TABLE 1
Moments, skewness and kurtosis for various choices of parameters.

Parameter	Raw moments	Central moments	Mode	Skewness	Kurtosis
$\theta=1.0$ $\beta=5.0$ $\lambda=-0.5$	$\mu_1'=0.25$ $\mu_2'=0.11$ $\mu_3'=0.07$ $\mu_4'=0.06$	$\mu_2=0.048$ $\mu_3=0.019$ $\mu_4=0.020$	0.058	3.28	8.66
$\theta=0.5$ $\beta=0.5$ $\lambda=-0.5$	$\mu_1'=4.61$ $\mu_2'=38.44$ $\mu_3'=467.34$ $\mu_4'=7509.17$	$\mu_2=17.19$ $\mu_3=131.66$ $\mu_4=2438.05$	1.114	3.414	8.25
$\theta=0.5$ $\beta=10$ $\lambda=-0.5$	$\mu_1'=0.231$ $\mu_2'=0.096$ $\mu_3'=0.058$ $\mu_4'=0.047$	$\mu_2=0.043$ $\mu_3=0.016$ $\mu_4=0.015$	0.055	3.35	8.58
$\theta=2.5$ $\beta=0.5$ $\lambda=-0.5$	$\mu_1'=1.14$ $\mu_2'=2.25$ $\mu_3'=6.19$ $\mu_4'=21.74$	$\mu_2=0.95$ $\mu_3=1.46$ $\mu_4=5.99$	0.076	2.477	6.63
$\theta=0.5$ $\beta=1.0$ $\lambda=-0.5$	$\mu_1'=2.31$ $\mu_2'=9.61$ $\mu_3'=58.42$ $\mu_4'=469.32$	$\mu_2=4.27$ $\mu_3=16.48$ $\mu_4=151.78$	0.557	3.48	8.31
$\theta=0.5$ $\beta=1.0$ $\lambda=0.5$	$\mu_1'=1.62$ $\mu_2'=5.95$ $\mu_3'=34.54$ $\mu_4'=273.16$	$\mu_2=3.33$ $\mu_3=14.13$ $\mu_4=122.37$	0	5.43	11.06
$\theta=3.0$ $\beta=2.0$ $\lambda=0.5$	$\mu_1'=0.116$ $\mu_2'=0.028$ $\mu_3'=0.011$ $\mu_4'=0.006$	$\mu_2=0.015$ $\mu_3=0.004$ $\mu_4=0.003$	0	6.23	12.35

4.5. Moment generating function

The moment generating function of ETE distribution is obtained as

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_0^\infty e^{tx} \sum_{k=0}^{\infty} S_k(\theta,\lambda)(k+\theta)\beta e^{-(k+\theta)\beta x} dx \\ &= \sum_{k=0}^{\infty} S_k(\theta,\lambda)(k+\theta)\beta \int_0^\infty e^{-[(k+\theta)\beta - t]x} dx \\ &= \sum_{k=0}^{\infty} S_k(\theta,\lambda)(k+\theta)\beta \frac{1}{(k+\theta)\beta - t}. \end{aligned} \tag{45}$$

4.6. Maximum likelihood estimation of the parameters

The likelihood function of ETE distribution is given by

$$L(x;\theta,\beta,\lambda) = (\theta\beta)^n e^{-\theta\beta\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left(1-\lambda+2\lambda e^{-\beta x_i}\right)\prod_{i=1}^{n}\left(1-\lambda+\lambda e^{-\beta x_i}\right)^{\theta-1}. \tag{46}$$

The log likelihood function is

$$\begin{aligned} \ln\big(L(x;\theta,\beta,\lambda)\big) &= n\ \ln(\theta) + n\ \ln(\beta) - \theta\beta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln(1-\lambda+2\lambda e^{-\beta x_i}) \\ &\quad +(\theta-1)\sum_{i=1}^{n}\ln(1-\lambda+\lambda e^{-\beta x_i}). \end{aligned} \tag{47}$$

Equation (47) can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating this with respect to θ, β and λ. The components of the score vector

$$V(\Theta) = \left(\frac{\partial\ \ln(L(x;\theta,\beta,\lambda))}{\partial\theta}, \frac{\partial\ \ln(L(x;\theta,\beta,\lambda))}{\partial\beta}, \frac{\partial\ \ln(L(x;\theta,\beta,\lambda))}{\partial\lambda}\right)$$

are given by

$$\begin{aligned} \frac{\partial\ \ln\big(L(x;\theta,\beta,\lambda)\big)}{\partial\theta} &= \frac{n}{\theta} - \beta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln(1-\lambda+\lambda e^{-\beta x_i}), \\ \frac{\partial\ \ln\big(L(x;\theta,\beta,\lambda)\big)}{\partial\beta} &= \frac{n}{\beta} - \theta\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\frac{2\lambda\beta e^{-\beta x_i}}{1-\lambda+2\lambda e^{-\beta x_i}} - \sum_{i=1}^{n}\frac{\lambda\beta(\theta-1)e^{-\beta x_i}}{1-\lambda+\lambda e^{-\beta x_i}}, \\ \frac{\partial\ \ln\big(L(x;\theta,\beta,\lambda)\big)}{\partial\lambda} &= \sum_{i=1}^{n}\frac{2e^{-\beta x_i}}{1-\lambda+2\lambda e^{-\beta x_i}} + (\theta-1)\sum_{i=1}^{n}\frac{e^{-\beta x_i}-1}{1-\lambda+\lambda e^{-\beta x_i}}. \end{aligned}$$

That is, the normal equations takes the following form

$$\frac{n}{\theta} - \beta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln(1 - \lambda + \lambda e^{-\beta x_i}) = 0, \tag{48}$$

$$\frac{n}{\beta} - \theta \sum_{i=1}^{n} x_i - 2\lambda\beta \sum_{i=1}^{n} \frac{e^{-\beta x_i}}{1 - \lambda + 2\lambda e^{-\beta x_i}} - \lambda\beta(\theta - 1) \sum_{i=1}^{n} \frac{e^{-\beta x_i}}{1 - \lambda + \lambda e^{-\beta x_i}} = 0, \tag{49}$$

$$\sum_{i=1}^{n} \frac{2e^{-\beta x_i}}{1 - \lambda + 2\lambda e^{-\beta x_i}} + (\theta - 1) \sum_{i=1}^{n} \frac{e^{-\beta x_i} - 1}{1 - \lambda + \lambda e^{-\beta x_i}} = 0. \tag{50}$$

These equations do not have explicit solutions and they have to be obtained numerically. From (48), the MLE of θ can be obtained as follows

$$\hat{\theta} = \frac{n}{\beta \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \ln(1 - \lambda + \lambda e^{-\beta x_i})}. \tag{51}$$

Substituting (51) in (49) and (50), we get the MLEs of β and λ. Statistical softwares like *nlm* package in R programming can be use to solve these equations numerically.

4.7. *Simulation study*

In order to check the performance of the maximum likelihood estimate given by (46) we conducted a simulation study.

TABLE 2

Average of MLEs with standard error of $ETE(\theta, \beta, \lambda)$ for various choices of parameter values.

$Parameters$	n	$\hat{\theta}(\hat{se}(\hat{\theta}))$	$\hat{\beta}(\hat{se}(\hat{\beta}))$	$\hat{\lambda}(\hat{se}(\hat{\lambda}))$
$\theta = 1$, $\beta = 5$, $\lambda = -0.5$	50	0.990(1.108)	5.717(1.891)	-0.598(0.275)
	100	1.112(1.081)	5.228(1.840)	-0.595(0.299)
	200	1.049(0.717)	5.114(1.948)	-0.579(0.138)
	500	1.047(0.667)	5.007(1.901)	-0.554(0.101)
$\theta = 5$, $\beta = 10$, $\lambda = 0.5$	50	4.416(1.231)	9.639(2.054)	0.622(0.342)
	100	4.695(0.843)	9.747(1.479)	0.581(0.189)
	200	4.831(1.207)	9.839(1.582)	0.513(0.221)
	500	4.919(0.943)	10.098(1.809)	0.507(0.226)
$\theta = 0.5$, $\beta = 0.5$, $\lambda = -0.9$	50	0.418(0.299)	0.431(0.238)	-0.807(0.167)
	100	0.432(0.257)	0.452(0.370)	-0.814(0.121)
	200	0.456(0.248)	0.475(0.248)	-0.851(0.093)
	500	0.499(0.201)	0.507(0.164)	-0.883(0.071)
$\theta = 10$, $\beta = 2.5$, $\lambda = 0.8$	50	9.120(1.155)	2.249(0.792)	0.869(0.149)
	100	9.435(0.827)	2.496(0.755)	0.837(0.153)
	200	9.971(0.729)	2.507(0.479)	0.825(0.071)
	500	10.094(0.226)	2.503(0.223)	0.818(0.039)

We take the sample sizes to be $n = 50, 100, 200$ and 500. The process is replicated 1000 times and the average estimate along with the standard error are presented in Table 2. Here we can see that as the sample size increases the MLE estimates of ETE distribution converges to the true value and the corresponding standard error decreases.

4.8. Entropies

Entropy measures the variation or uncertainty of a random variable X. The popular measures of entropy are Rényi and Shannon entropies. The Rényi entropy of a random variable X, with pdf $j(x)$ is defined as,

$$I_R(\gamma) = \frac{1}{1-\gamma} \ln\Big(\int_0^\infty j^\gamma(x) dx \Big),$$

for $\gamma > 0$ and $\gamma \neq 1$, see Rényi (1961). The Rényi entropy for a random variable from the T-transmuted X family of distributions is obtained as

$$I_R(\gamma) = \frac{1}{1-\gamma} \ln\Big[\int_0^\infty \frac{g^\gamma(x)[1+\lambda-2\lambda G(x)]^\gamma}{[1-G(x)(1+\lambda\bar{G}(x))]^\gamma} \big(r\{-\ln(1-G(x)[1+\lambda\bar{G}(x)])\}\big)^\gamma dx \Big], \tag{52}$$

where $G(x)$ is the cdf of the base distribution with pdf $g(x)$ and $r\{.\}$ is the pdf of an arbitrary distribution.

For the ETE distribution, the Rényi entropy is obtained by using (52) as

$$\begin{aligned} I_R(\gamma) &= \frac{\gamma}{1-\gamma}\Big(\ln(\theta) + \ln(\beta) \Big) \\ &\quad + \frac{\gamma}{1-\gamma} \ln\Big(\int_0^\infty e^{-\gamma\theta\beta x}\big(1-\lambda+\lambda e^{-\beta x}\big)^{\gamma(\theta-1)} \\ &\quad \big(1-\lambda+2\lambda e^{-\beta x}\big)^\gamma dx \Big). \end{aligned} \tag{53}$$

For given values of θ, β, λ and γ, the Rényi entropy can be numerically computed using R programming or any other statical softwares. Table 3 shows the values of entropy for given parameter values and γ.

The Shannon entropy of a random variable X is defined by $E\big(-\ln[j(X)]\big)$, see Shannon (1948). It is the special case of the Rényi entropy when $\gamma \to 1$. The Shannon entropy of T-transmuted X family of distributions is given by

$$E\Big(-\ln\big(j(X)\big)\Big) = -E\Big(\ln\big(g(X)\big)\Big) - E\Big(\ln\big[1+\lambda-2\lambda G(X)\big]\Big)$$
$$+E\Big(\ln\big[1-G(x)\big(1+\lambda\bar{G}(x)\big)\big]\Big)$$
$$+E\Big[\ln\big[\gamma\big(-\ln\big[1-G(X)\big(1+\lambda\bar{G}(X)\big)\big]\big)\big]\Big]. \quad (54)$$

For the ETE distribution, the Shannon entropy obtained as

$$E\Big(-\ln\big(j(X)\big)\Big) = \theta\beta E(X) - E\Big[\ln\Big(1-\lambda+\lambda\exp^{-\beta X}\Big)\Big]$$
$$-(\theta-1)E\Big[\ln\Big(1-\lambda+\lambda\exp^{-\beta X}\Big)\Big]. \quad (55)$$

TABLE 3
Rényi entropy for given values of θ, β, λ and γ.

θ	β	λ	$\gamma=0.5$	$\gamma=2.0$	$\gamma=3.0$	$\gamma=5.0$
1	1	-1	1.5963	1.0986	1.0075	0.9183
		-0.5	1.5216	0.9808	0.8836	0.7907
		0.5	1.1732	0.3449	0.1813	0.0189
		1	0.6931	0.0000	-0.1438	-0.2908
2	1	-1	1.0383	0.5935	0.5106	0.4284
		-0.5	0.9081	0.6768	0.2939	0.1999
		0.5	0.4029	-0.3769	-0.5306	-0.6852
		1	0.0001	-0.6931	-0.8369	-0.9835
2	2	-1	0.3452	-0.0996	1.4407	-0.1826
		-0.5	0.2149	-0.3028	-0.3993	-0.4933
		0.5	-0.2903	-1.0700	-1.2237	-1.3783
		1	-0.6932	-1.3863	-1.5301	-1.6771
0.5	0.5	-1	2.8918	2.3375	2.2357	2.1371
		-0.5	2.8510	2.2723	2.1683	2.0696
		0.5	1.9387	1.7819	1.6028	1.4261
		1	2.0794	1.3863	1.2427	1.0955

5. Applications

In this section, to show how the ETE distribution works in practice, we use two real data sets. We compare the fit of the ETE distribution with the following life time distributions:

(a) Kumaraswamy Exponential (KuE) distribution having pdf

$$f(x;\theta,\beta,c)=\theta\beta c e^{-cx}(1-e^{-cx})^{\theta-1}[1-(1-e^{-cx})^{\theta}]^{\beta-1};\ x>0,\theta,\beta,c>0.$$

(b) Exponentiated Weibull (EW) distribution having pdf

$$f(x;\theta,\beta,c)=\theta\beta^{\theta}c x^{\theta-1}e^{-(\beta x)^{\theta}}(1-e^{-(\beta x)^{\theta}})^{c-1};\ x>0,\theta,\beta,c>0.$$

(c) Weibull (W) distribution having pdf

$$f(x;\theta,\beta)=\beta\theta^{\beta}x^{\beta-1}e^{-(\theta x)^{\beta}};\ x>0,\theta,\beta>0.$$

(d) Exponential (E) distribution having pdf

$$f(x;\theta)=\theta e^{-\theta x};\ x>0,\theta>0.$$

The values of the log-likelihood functions ($-\ln(L)$), the statistic $K-S$ (Kolmogorov-Smirnov), AIC (Akaike Information Criterion), $AICC$ (Akaike Information Criterion with correction) and BIC (Bayesian Information Criterion) are calculated for the five distributions in order to verify which distribution fits better to these data. The better distribution corresponds to smaller $K-S$, $-\ln(L)$, AIC, $AICC$ and BIC values and high p-value. Here, $AIC=-2\ \ln(L)+2k$, $AICC=-2\ \ln(L)+(\frac{2kn}{n-k-1})$ and $BIC=-2\ \ln(L)+k\ \ln(n)$ where L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size. The $K-S$ distance $D_n=\sup_x|F(x)-F_n(x)|$, where, $F_n(x)$ is the empirical distribution function.

5.1. First data set

Here we consider the data set of the life of fatigue of Kelvar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. The data sets are taken from Andrews and Herzberg (1985). The data are: (0.0251, 0.6751, 1.0483, 1.4880, 1.8808, 2.2460, 3.4846, 0.0886, 0.6753, 1.0596, 1.5728, 1.8878, 2.2878, 3.7433, 0.0891, 0.7696, 1.0773, 1.5733, 1.8881, 2.3203, 3.7455, 0.2501, 0.8375, 1.1733 1.7083, 1.9316, 2.3470, 3.9143, 0.3113, 0.8391, 1.2570, 1.7263, 1.9558, 2.3513, 4.8073, 0.3451, 0.8425, 1.2766, 1.7460, 2.0048, 2.4951, 5.4005, 0.4763, 0.8645, 1.2985, 1.7630, 2.0408, 2.5260, 5.4435, 0.5650, 0.8851, 1.3211, 1.7746, 2.0903, 2.9941, 5.5295, 0.5671, 0.9113, 1.3503, 1.8275, 2.1093, 3.0256, 6.5541, 0.6566, 0.9120, 1.3551, 1.8375, 2.1330, 3.2678, 9.0960, 0.6748, 0.9836, 1.4595, 1.8503, 2.2100, 3.4045).

The descriptive statistics of the above data set are given in Table 4. The values in Table 5 shows that the ETE distribution leads to a better fit compared to the other four models.

TABLE 4
Descriptive statistics of first data set.

n	Min	Max	Mean	Median	Sd	Skewness	Kurtosis
76	0.0251	9.0960	1.959	1.736	1.57	2.019	5.60

TABLE 5
Parameter estimates and goodness of fit statistics for various models fitted for the first data set.

Model	ML estimates	$-\ln(L)$	AIC	AICC	BIC	K-S	p-value
ETE	$\hat{\theta}=1.346$ $\hat{\beta}=0.579$ $\hat{\lambda}=-0.848$	121.461	248.922	249.255	255.914	0.0984	0.4266
KuE	$\hat{\theta}=1.556$ $\hat{\beta}=2.448$ $\hat{c}=0.328$	122.094	250.188	250.521	257.180	0.0990	0.4191
EW	$\hat{\theta}=1.101$ $\hat{\beta}=0.609$ $\hat{c}=1.443$	122.166	250.332	250.665	257.324	0.0992	0.4160
W	$\hat{\beta}=1.326$ $\hat{\theta}=0.469$	122.526	249.052	249.216	253.714	0.1098	0.2968
E	$\hat{\theta}=0.510$	127.114	256.228	256.282	258.559	0.5120	0.0266

The inverse of the Hessian matrix of the MLEs of ETE distribution is computed as

$$\begin{pmatrix} 1.53140 & -0.49821 & 0.00369 \\ -0.49821 & 0.16548 & -0.00493 \\ 0.00369 & -0.00493 & 0.01661 \end{pmatrix}$$

The 95% confidence interval for θ, β and λ are $(1.0678, 1.6242)$,$(0.4875, 0.6705)$ and $(-0.8770, -0.8190)$ respectively.

Figure 7 shows the fitted density curves for the first data set.

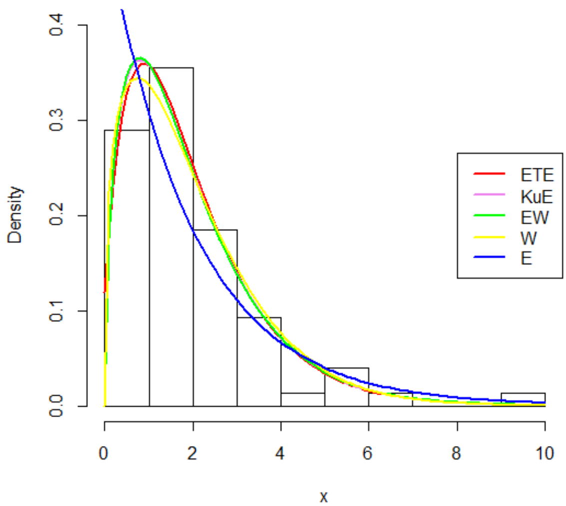

Figure 7 – Fitted pdf plots of first data set.

5.2. Second data set

The second real data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 taken from Lee (1992). The data are: (0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3,11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0).

TABLE 6
Descriptive statistics of second data set.

n	Min	Max	Mean	Median	Sd	Skewness	Kurtosis
121	0.30	154	46.33	40.00	35.28	1.056	0.471

The descriptive statistics of the second data set are given in Table 6.

The values in Table 7 shows that the ETE distribution leads to a better fit compared to the other four models.

TABLE 7
Parameter estimates and goodness of fit statistics for various models fitted for the second data set.

Model	ML estimates	$-\ln(L)$	AIC	AICC	BIC	K-S	p-value
ETE	$\hat{\theta}=1.876$ $\hat{\beta}=0.018$ $\hat{\lambda}=-0.765$	578.878	1163.76	1163.96	1172.14	0.0569	0.8284
KuE	$\hat{\theta}=1.651$ $\hat{\beta}=0.098$ $\hat{c}=0.231$	583.314	1172.63	1172.83	1181.02	0.1152	0.0803
EW	$\hat{\theta}=1.393$ $\hat{\beta}=0.017$ $\hat{c}=0.798$	579.879	1165.76	1165.96	1174.15	0.0664	0.6606
W	$\hat{\beta}=1.3056$ $\hat{\theta}=0.0199$	580.024	1164.05	1164.15	1169.64	0.0588	0.7967
E	$\hat{\theta}=0.022$	585.128	1172.26	1172.29	1175.05	0.1206	0.0594

The inverse of the Hessian matrix of the MLEs of ETE distribution is computed as

$$\begin{pmatrix} 0.18112 & 0.02364 & 0.03916 \\ 0.02364 & 0.00017 & 0.00015 \\ 0.03916 & 0.00015 & 0.01529 \end{pmatrix}$$

The 95% confidence interval for θ, β and λ are $(1.8002, 1.9518)$,$(0.0157, 0.0203)$ and $(-0.7870, -0.7430)$ respectively.

Figure 8 shows the fitted density curves for the second data set.

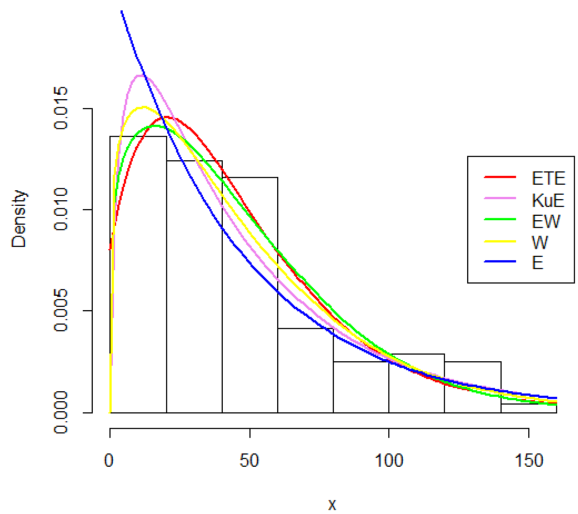

Figure 8 – Fitted pdf plots of second data set.

6. Conclusion

Transmuted family of distributions and $T-X$ family of distributions have been of great attention among the researchers recently. In this paper we introduce a new family, namely, "T-transmuted X family" of distributions. Many of the existing family of distributions are sub models of this new family. A special case of this family, exponential-transmuted exponential (ETE) distribution is studied in detail. Two real data sets are analyzed to show the importance and flexibility of this distribution.

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SUMMARY

Using the quadratic transmutation map ($QRTM$) approach of Shaw and Buckley (2007) and the $T-X$ family method by Alzaatreh *et al.* (2013b), we have developed a new family of distributions called T-transmuted X family of distributions. Many of the existing family of distributions are sub models of this family. As a special case, exponential transmuted exponential (ETE) distribution is studied in detail. The application and flexibility of this new distribution is illustrated using two real data sets.

Keywords: Exponential distribution; Hazard rate; Maximum likelihood estimation; Moments; T-X family of distributions.