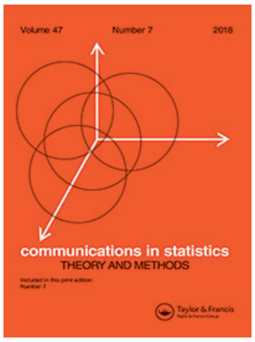

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Discrete Weibull geometric distribution and its properties

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ABSTRACT
In this article, the discrete analog of Weibull geometric distribution is introduced. Discrete Weibull, discrete Rayleigh, and geometric distributions are submodels of this distribution. Some basic distributional properties, hazard function, random number generation, moments, and order statistics of this new discrete distribution are studied. Estimation of the parameters are done using maximum likelihood method. The applications of the distribution is established using two datasets.

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1. Introduction

In lifetime modeling, the observed measurements are usually discrete in nature, because they are taken to only a finite value and cannot really constitute all points in a continuum. We come across such situations where lifetimes are measured on a discrete scale. For example, the curing time of a particular disease measured in days, the survival time of cancer patients in months etc. (see Krishna and Singh 2009). Here, the continuous lifetime may not necessarily always be measured on a continuous scale, but may often be counted as discrete random variable.

Developing a discrete version of continuous distributions has drawn attention of the researchers. In recent decades, a large number of research papers dealing with discrete distribution derived by discretizing continuous random variables have appeared in the statistics literature. Lisman and van Zuylen (1972) proposed and Kemp (1997) studied the discrete normal distribution. Salvia and Bollinger (1982) introduced the basic results about discrete reliability and illustrated them with discrete lifetime distributions with one parameter. Roy (2003) studied another version of the discrete normal distribution. A discrete analog of Weibull distribution was first proposed by Nakagawa and Osaki (1975). Stein and Dattero (1984) introduced a second type of discrete Weibull (DW) distribution and a third one was proposed by Padgett and Spurrier (1985). Sato et al. (1999) proposed a discrete exponential distribution and applied this distribution to model defect count in a semiconductor deposition equipment and defect count distribution per chips. Inusah and Kozubowski (2006) and Kozubowski and Inusah (2006) introduced discrete analog of Laplace and skew-Laplace

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distributions. Krishna and Pundir (2009) introduced the discrete Burr distribution which led to the discrete Pareto distribution. A discrete analog of the generalized exponential distribution of Gupta and Kundu (1999) was proposed by Nekoukhou, Alamatsaz, and Bidram (2012). Chakraborty and Chakravarty (2012) introduced the discrete gamma distribution.

The present article is organized as follows. In Section 2, we discuss the methods of discretizing a continuous distribution. In Section 3, the DW geometric distribution is introduced and in Section 4, various properties of this distribution are studied. Applications of this new distribution are discussed in Section 5. Here, we consider two datasets and is shown that for both the datasets, the DW geometric distribution is the appropriate model. Conclusions are presented in Section 6.

2. Discretization of a continuous distribution

Continuous random variable may be characterized by its probability density function (pdf), hazard rate function, moments, etc. Construction of a discrete analog from a continuous distribution is based on the principle of preserving one or more characteristic property of the continuous one. Discretization of continuous distribution can be done using different methodologies. Some of them are: (i) discretize the continuous cumulative distribution function (cdf), (ii) discretize the continuous pdf, (iii) discretize the continuous hazard rate function, and (iv) obtain discrete lifetime distribution from the alternative hazard rate. A detailed survey of the methods and constructions of discrete analogs of continuous distributions are discussed in Chakraborty (2015).

Let X be a continuous random variable, then its discrete analog Y can be derived by using the survival function as follows:

$$P(Y=y)=P(X\geq y)-P(X\geq y+1)=S_X(y)-S_X(y+1);\quad y=0,1,2,\ldots. \tag{1}$$

where $Y=\lfloor X\rfloor=$ largest integer less than or equal to X and $S_X(.)$ is the survival function of the random variable X. For a given continuous distribution, it is possible to generate the corresponding discrete distribution using the formula (1) above. Suppose the underlying distribution is exponential with survival function, $S_X(x)=P(X\geq x)=e^{-\lambda x}$, then the probability mass function (pmf) of its discrete version is given by

$$P(Y=y)=e^{-\lambda y}-e^{-\lambda(y+1)}=q^y-q^{y+1}=(1-q)q^y;\quad y=0,1,2,\ldots, \tag{2}$$

where $q=e^{-\lambda}$. This is the geometric distribution with parameter q. By the similar way, Nakagawa and Osaki (1975) proposed the DW distribution with pmf,

$$P(Y=y)=q^{y^\beta}-q^{(y+1)^\beta};\quad y=0,1,2,\ldots.;\quad \beta>0;\quad 0<q<1, \tag{3}$$

where $q=e^{-\lambda}$.

3. Discrete Weibull geometric distribution

Suppose that $\{X_1,X_2,\ldots,X_n\}$ are independent and identically distributed (iid) random variables following the Weibull distribution, $W(\beta,\alpha)$ with scale parameter $\beta>0$, shape parameter $\alpha>0$ and pdf,

$$g(x;\beta,\alpha)=\alpha\beta^\alpha x^{\alpha-1}e^{-(\beta x)^\alpha},\ x>0 \tag{4}$$

and let N be a discrete random variable having geometric distribution with pmf, $P(n;p)=(1-p)p^{n-1}$ for $n\in\mathbf{N}$ and $p\in(0,1)$.

Let $X_{(1)} = Min\{X_i\}_{i=1}^{N}$. The conditional cumulative distribution of $X_{(1)} \mid N = n$ is given by,

$$G_{\{X_{(1)}|N=n\}} = 1 - [1 - F(x)]^n = 1 - e^{-n(\beta x)^\alpha}. \tag{5}$$

The cdf of $X_{(1)}$ is given by

$$\begin{aligned} F(x; p, \beta, \alpha) &= (1-p)\sum_{n=1}^{\infty} p^{n-1}[1 - e^{-n(\beta x)^\alpha}] \\ &= \frac{1 - e^{-(\beta x)^\alpha}}{1 - pe^{-(\beta x)^\alpha}}; \quad x > 0,\ 0 < p < 1,\ \beta > 0, \quad \alpha > 0. \end{aligned} \tag{6}$$

The marginal pdf of $X_{(1)}$ is

$$f(x; p, \beta, \alpha) = \alpha\beta^\alpha(1-p)x^{\alpha-1}e^{-(\beta x)^\alpha}(1 - pe^{-(\beta x)^\alpha})^{-2}; \quad x > 0. \tag{7}$$

The distribution of $X_{(1)}$ is called Weibull geometric and is denoted by $WG(p, \beta, \alpha)$. This distribution is studied by Barreto-Souza, de Morais, and Cordeiro (2011). The survival and hazard rate functions are

$$S(x; p, \beta, \alpha) = \frac{(1-p)e^{-(\beta x)^\alpha}}{1 - pe^{-(\beta x)^\alpha}}; \quad x > 0, \tag{8}$$

and

$$h(x; p, \beta, \alpha) = \frac{\alpha\beta^\alpha x^{\alpha-1}}{1 - pe^{-(\beta x)^\alpha}}; \quad x > 0, \tag{9}$$

respectively.

Using the method (1), after the reparametrization $\rho = e^{-\beta^\alpha}$, the pmf of the discrete version Y of the Weibull geometric distribution is derived as

$$p_Y(y; p, \rho, \alpha) = P(Y = y) = S_X(y) - S_X(y+1) = \frac{(1-p)(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})}{(1 - p\rho^{y^\alpha})(1 - p\rho^{(y+1)^\alpha})}, \tag{10}$$

where $y = 0, 1, 2, \ldots;\ \alpha > 0,\ 0 < p < 1$ and $0 < \rho < 1$. We call this distribution as DW geometric distribution with parameters $p,\ \rho$, and α and is denoted as $DWG(p, \rho, \alpha)$.

Here note that, $\sum_{y=0}^{\infty} P(Y = y) = S_X(0) - S_X(1) + S_X(1) - S_X(2) + \ldots. = S_X(0) = 1$.

In particular, when $\alpha = 1$, the pmf becomes,

$$p_Y(y; p, \rho) = \frac{(1-p)(\rho^y - \rho^{(y+1)})}{(1 - p\rho^y)(1 - p\rho^{(y+1)})}.$$

This is the pmf of the discrete exponential geometric distribution.

When $p \to 0$, we get $p_Y(y; \rho, \alpha) = \rho^{y^\alpha} - \rho^{(y+1)^\alpha}$, which is the DW distribution (Nakagawa and Osaki 1975) with parameters ρ and α.

When $p \to 0$ and $\alpha \to 2$, then, $p_Y(y; \rho) = \rho^{y^2} - \rho^{(y+1)^2}$, which is the discrete Rayleigh distribution (Roy 2004).

When $p \to 0$ and $\alpha \to 1$, then, $p_Y(y; \rho) = \rho^y - \rho^{(y+1)}$, which is the geometric distribution with parameter ρ.

4. Structural properties of $DWG(p, \rho, \alpha)$

The pmf plots of $DWG(p, \rho, \alpha)$ for various choices of the values of the parameters have been presented in Figure 1.

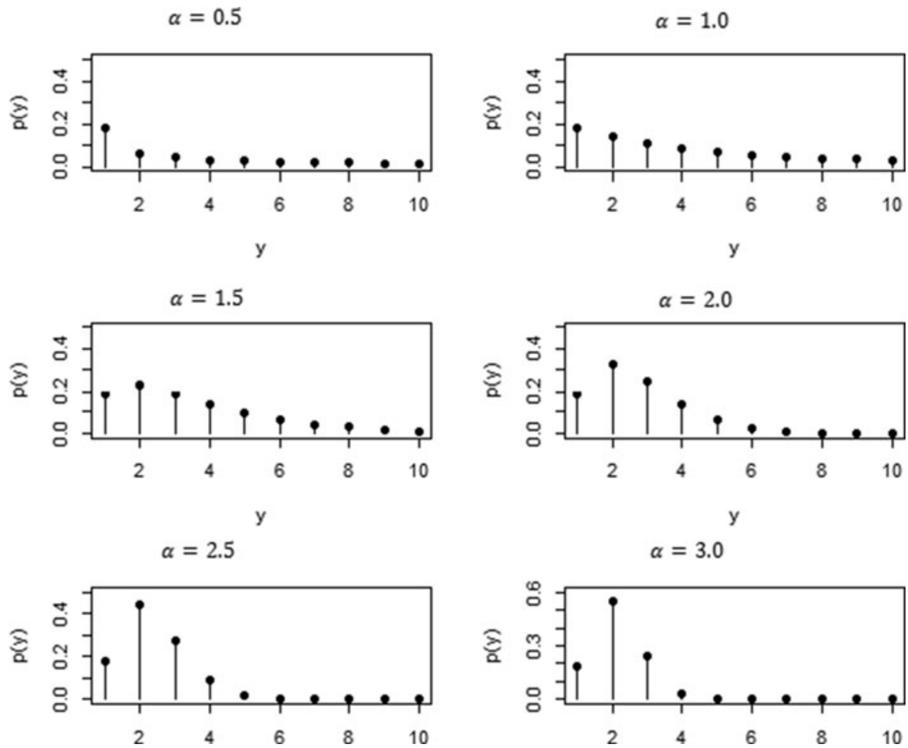

Figure 1. Plots of the pmf of $DWG(p, \rho, \alpha)$ for $p = 0.5$, $\rho = 0.9$, and $\alpha = (0.5, 1.0, 1.5, 2.0, 2.5, 3.0)$.

4.1. Recurrence relation for probabilities

The probabilities can be calculated recursively using the following relation :

$$p_Y(y+1; p, \rho, \alpha) = \frac{(1 - p\rho^{y^\alpha})(\rho^{(y+1)^\alpha} - \rho^{(y+2)^\alpha})}{(1 - p\rho^{(y+2)^\alpha})(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})} p_Y(y; p, \rho, \alpha). \tag{11}$$

Gupta, Gupta, and Tripathi (1997) proposed analogous statements for discrete distributions with unbounded support as:

(a) The distribution is log-concave if and only if $\{\frac{p_Y(y+1)}{p_Y(y)}\}_{y\geq 0}$ is decreasing. Then, the hazard rate is increasing.
(b) The distribution is log-convex if and only if $\{\frac{p_Y(y+1)}{p_Y(y)}\}_{y\geq 0}$ is increasing. Then, the hazard rate is decreasing.
(c) If the sequence $\{\frac{p_Y(y+1)}{p_Y(y)}\}_{y\geq 0}$ is constant, the hazard rate is constant and the distribution is geometric.

For the $DWG(p, \rho, \alpha)$,

$$\frac{p(y+1)}{p(y)} = \frac{(1 - p\rho^{y^\alpha})(\rho^{(y+1)^\alpha} - \rho^{(y+2)^\alpha})}{(1 - p\rho^{(y+2)^\alpha})(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})}.$$

Let $\delta(y) = 1 - \frac{p(y+1)}{p(y)}$ and $\Delta\delta(y) = \delta(y+1) - \delta(y)$. Then,

(a) If $\Delta\delta(y) > 0$ (log-concavity), $h(y)$ is increasing.
(b) If $\Delta\delta(y) < 0$ (log-convexity), $h(y)$ is decreasing.
(c) If $\Delta\delta(y) = 0$, $h(y)$ is constant hazard rate, where $h(y)$ is the hazard rate function.

4.2. Cumulative distribution function

The cdf of $DWG(p, \rho, \alpha)$ is obtained as

$$F(y) = P(Y \leq y) = 1 - S_X(y) + P(Y = y) = \frac{1 - \rho^{(y+1)^\alpha}}{1 - p\rho^{(y+1)^\alpha}} \tag{12}$$

where $y = 0, 1, 2, \ldots;\ 0 < p < 1,\ 0 < \rho < 1$ and $\alpha > 0$.

Here note that,

$$F(0) = \frac{1 - \rho}{1 - p\rho}.$$

The proportion of positive values, $1 - F(0) = \frac{\rho(1-p)}{1-p\rho}$.

Also,

$$P(a < Y \leq b) = \frac{1 - \rho^{(b+1)^\alpha}}{1 - p\rho^{(b+1)^\alpha}} - \frac{1 - \rho^{(a+1)^\alpha}}{1 - p\rho^{(a+1)^\alpha}}.$$

4.3. Survival and hazard rate functions

The survival function of $DWG(p, \rho, \alpha)$ is given by

$$S(y) = P(Y > y) = 1 - P(Y \leq y) = \frac{(1 - p)\rho^{(y+1)^\alpha}}{1 - p\rho^{(y+1)^\alpha}}. \tag{13}$$

Discrete hazard rates may applicable in several common situations in reliability theory and survival analysis where clock time is not the best scale on which to describe the lifetime. For example, in weapons like tanks, the number of rounds fired until failure is more important than lifetime in failure. In other situations, a device is monitored only once per time period and the observation then is the number of time periods successfully completed prior to the failure of the device. Similarly, in survival analysis, one may be interested in the length of stay (usually measured as the number of days) in an observation ward or survival time (measured in number of weeks) of leukemia patients. In all these cases, the lifetimes are not measured on continuous scale but are simply counted and hence are discrete random variables. For the application of the hazard rate functions to characterizations of aging properties of discrete lifetimes distributions, one can see Shaked, Shanthikumar, and Valdez-Torres (1995).

The hazard rate function of $DWG(p, \rho, \alpha)$ is given by,

$$h(y) = P(Y = y/Y \geq y) = \frac{P(Y = y)}{P(Y \geq y)} = \frac{1 - \rho^{(y+1)^\alpha - y^\alpha}}{1 - p\rho^{(y+1)^\alpha}}, \tag{14}$$

provided $P(Y \geq y) > 0$. It indicates the conditional probability of the system at time y, given that it did not fail before time y.

When $y \to 0$, here note that from (14),

$$h(y) \to \frac{1 - \rho}{1 - p\rho} = p_Y(0).$$

For $\alpha = 1$,

$$h(y) = \frac{1 - \rho}{1 - p\rho^{(y+1)}}.$$

Also note that, as $y \to \infty$, $h(y) \to 1 - \rho$.

We have

$$h(0) = \frac{1-\rho}{1-p\rho},$$

$$h(1) = \frac{1-\rho}{1-p\rho^2},$$

$$h(2) = \frac{1-\rho}{1-p\rho^3}, \ldots.$$

That is, $h(0) > h(1) > h(2) > \ldots$. Therefore, $h(y)$ is decreasing from $\frac{1-\rho}{1-p\rho}$ to $1-\rho$.

Also let,

$$h(y) = \frac{1-\rho}{1-p\rho^{y+1}} = m$$

where, m is constant, such that $1-\rho < m < \frac{1-\rho}{1-p\rho}$. Then the value of y, corresponds to m is obtained as,

$$y = \frac{\ln(m+\rho-1) - \ln(p) - \ln(m)}{\ln(\rho)} - 1.$$

Since, y is discrete, we may take the floor value of y.

Then, for $0 < \alpha < 1$, as $y \to \infty$, $h(y) \to 0$. In this case, the hazard rate function $h(y)$ is decreasing from $\frac{1-\rho}{1-p\rho}$ to 0 .

For $\alpha > 1$, $h(y)$ is an increasing failure rate function (IFR).

The variation of hazard rate function for the increase in α values with the given values of p and ρ is shown in Table 1.

The accumulated hazard function, $H(y)$, is given by,

$$H(y) = \sum_{t=0}^{y} h(t) = \sum_{t=0}^{y} \frac{1-\rho^{(t+1)^\alpha - t^\alpha}}{1-p\rho^{(t+1)^\alpha}}. \tag{15}$$

Table 1. Values of hazard rate function for $p = 0.5$, $\rho = 0.5$ and various choices of α.

y	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3$
0	0.667	0.667	0.667	0.667	0.667	0.667
1	0.307	0.571	0.773	0.903	0.970	0.994
2	0.233	0.533	0.817	0.970	0.999	0.999
3	0.194	0.516	0.858	0.992	0.999	0.999
4	0.169	0.508	0.890	0.998	0.999	0.999
5	0.151	0.504	0.913	0.999	0.999	0.999
6	0.138	0.502	0.929	0.999	0.999	0.999
7	0.128	0.501	0.942	0.999	0.999	0.999
8	0.120	0.500	0.952	0.999	0.999	0.999
9	0.113	0.500	0.959	0.999	0.999	0.999
10	0.107	0.500	0.966	0.999	0.999	0.999
11	0.102	0.500	0.971	0.999	0.999	0.999
12	0.097	0.500	0.975	0.999	0.999	0.999
13	0.094	0.500	0.978	0.999	0.999	0.999
14	0.090	0.500	0.981	0.999	0.999	0.999
15	0.087	0.500	0.983	0.999	0.999	0.999
16	0.084	0.500	0.985	0.999	0.999	0.999
17	0.082	0.500	0.987	0.999	0.999	0.999
18	0.079	0.500	0.989	0.999	0.999	0.999
19	0.077	0.500	0.990	0.999	0.999	0.999
20	0.075	0.500	0.991	0.999	0.999	0.999

The mean residual life (MRL) function is given by

$$L(y) = E[(Y-y)|Y \geq y] = \frac{\sum_{j\geq y} jp(j)}{\sum_{j\geq y} p(j)} - y = \frac{\sum_{j>y} S(j)}{S(y)} = \sum_{j\geq y}\prod_{i=y}^{j}(1-h(i))$$

$$= \sum_{j\geq y}\prod_{i=y}^{j}\frac{\rho^{(i+1)^\alpha}(1-p\rho^{i^\alpha})}{\rho^{i^\alpha}(1-p\rho^{(i+1)^\alpha})}; \quad y \geq 0.$$

Also, from Roy and Gupta (1999),

$$\mu(y) = E[(Y-y)|Y > y] = L(y+1)+1 = \sum_{j\geq y+1}\prod_{i=y+1}^{j}\frac{\rho^{(i+1)^\alpha}(1-p\rho^{i^\alpha})}{\rho^{i^\alpha}(1-p\rho^{(i+1)^\alpha})} + 1; \quad y > 0.$$

Assume that the MRL function at time $y = 0$ is equal to the mean of the lifetime distribution, that is, $L(0) = \mu$.

Then,

$$\mu(0) = \frac{\mu}{1-p(0)} = \frac{\mu(1-p\rho)}{\rho(1-p)}.$$

Figure 2 shows the shape of the hazard rate function for different choice of parameter values.

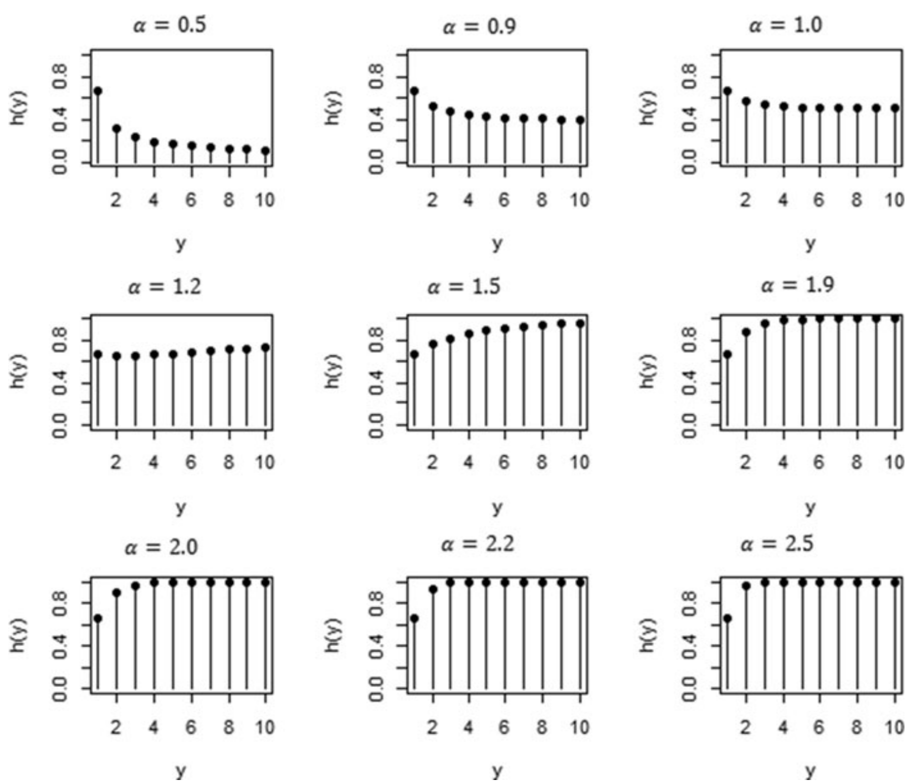

Figure 2. Shapes of hazard rate function for $p = 0.5$, $\rho = 0.5$, and various values of α.

The reverse hazard rate function is given by

$$h^*(y) = P(Y = y/Y \leq y) = \frac{P(Y = y)}{P(Y \leq y)} = \frac{(1-p)(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})}{(1 - p\rho^{y^\alpha})(1 - \rho^{(y+1)^\alpha})}. \tag{16}$$

The second rate of failure is given by

$$h^{**}(y) = \log\left\{\frac{S(y)}{S(y+1)}\right\} = \log\left\{\frac{(\frac{1}{\rho})^{(y+2)^\alpha} - p}{(\frac{1}{\rho})^{(y+1)^\alpha} - p}\right\}. \tag{17}$$

4.4. Quantiles and random number generation

From Rohatgi and Saleh (2001), the point y_u is known as the uth quantile of a discrete random variable Y, if it satisfies $P(Y \leq y_u) \geq u$ and $P(Y \geq y_u) \geq 1 - u$. Using this result, we have the following theorem.

Theorem 1. *The uth quantile $\phi(u)$ of $DWG(p, \rho, \alpha)$ is given by,*

$$\phi(u) = \lceil y_u \rceil = \left\lceil \left(\ln\left(\frac{u-1}{up-1}\right) \Big/ \ln(\rho)\right)^{\frac{1}{\alpha}} - 1 \right\rceil, \tag{18}$$

where $\lceil y_u \rceil$ denotes the smallest integer greater than or equal to y_u.

Proof. First suppose that, $P(Y \leq y_u) \geq u$.

That is,

$$\begin{aligned}
\frac{1 - \rho^{(y_u+1)^\alpha}}{1 - p\rho^{(y_u+1)^\alpha}} &\geq u \\
&\Longrightarrow 1 - \rho^{(y_u+1)^\alpha} \geq u(1 - p\rho^{(y_u+1)^\alpha}) \\
&\Longrightarrow \ln\left(\frac{1-u}{1-up}\right) \geq (y_u+1)^\alpha \ln(\rho) \\
&\Longrightarrow y_u \geq \left[\ln\left(\frac{1-u}{1-up}\right) \Big/ \ln(\rho)\right]^{\frac{1}{\alpha}} - 1,
\end{aligned} \tag{19}$$

since $\ln(\rho) < 0$.

Similarly, $P(Y \geq y_u) \geq 1 - u$ gives,

$$y_u \leq \left[\ln\left(\frac{1-u}{1-up}\right) \Big/ \ln(\rho)\right]^{\frac{1}{\alpha}}. \tag{20}$$

Combining (19) and (20),

$$\left[\ln\left(\frac{1-u}{1-up}\right) \Big/ \ln(\rho)\right]^{\frac{1}{\alpha}} \geq y_u > \left[\ln\left(\frac{1-u}{1-up}\right) \Big/ \ln(\rho)\right]^{\frac{1}{\alpha}} - 1.$$

Hence, $\phi(u)$ is an integer given by,

$$\phi(u) = \lceil y_u \rceil = \left\lceil \left(\ln\left(\frac{1-u}{1-up}\right) \Big/ \ln(\rho)\right)^{\frac{1}{\alpha}} - 1 \right\rceil.$$

This completes the proof. □

Table 2. MLEs of $DWG(p, \rho, \alpha)$ for various samples (n).

Parameters	n	$\hat{p}(\hat{SE}(\hat{p}))$	$\hat{\rho}(\hat{SE}(\hat{\rho}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$
$p = 0.5$	50	0.539 (2.781)	0.553 (1.498)	0.557 (0.915)
$\rho = 0.5$	100	0.581 (0.901)	0.554 (0.696)	0.485 (0.391)
$\alpha = 0.5$	500	0.599 (0.794)	0.534 (0.415)	0.501 (0.272)
	1000	0.491 (0.824)	0.535 (0.404)	0.533 (0.234)
$p = 0.75$	50	0.735 (0.933)	0.458 (0.869)	1.017 (1.072)
$\rho = 0.5$	100	0.734 (0.779)	0.503 (0.730)	1.269 (1.062)
$\alpha = 1.0$	500	0.805 (0.363)	0.553 (0.459)	1.104 (0.553)
	1000	0.751 (0.294)	0.531 (0.349)	1.026 (0.392)
$p = 0.6$	50	0.543 (1.709)	0.887 (0.411)	1.359 (1.065)
$\rho = 0.9$	100	0.709 (0.531)	0.913 (0.156)	1.556 (0.564)
$\alpha = 1.5$	500	0.638 (0.343)	0.907 (0.087)	1.549 (0.294)
	1000	0.623 (0.259)	0.908 (0.063)	1.527 (0.211)
$p = 0.9$	50	0.906 (0.383)	0.481 (1.010)	1.455 (1.746)
$\rho = 0.5$	100	0.936(0.249)	0.634 (0.899)	2.197 (2.230)
$\alpha = 2.0$	500	0.938 (0.103)	0.614 (0.393)	2.094 (0.925)
	1000	0.902(0.069)	0.637 (0.276)	2.012 (0.691)
$p = 0.8$	50	0.862 (0.895)	0.673 (1.434)	2.413 (3.940)
$\rho = 0.6$	100	0.858(0.687)	0.701 (1.012)	2.580 (3.038)
$\alpha = 2.5$	500	0.753 (0.565)	0.541 (0.568)	2.607 (1.621)
	1000	0.808 (0.145)	0.633 (0.278)	2.562 (0.965)

Using the usual inverse transformation method, a random number (integer) can be sampled from the proposed model . Let U be a random number drawn from a uniform distribution on (0, 1), then a random number Y following $DWG(p, \rho, \alpha)$ distribution is obtained by the Equation (18).

In particular, the median is given by

$$\phi(0.5) = \lceil y_{0.5} \rceil = \left\lceil \left(\ln\left(\frac{1}{2-p} \right) \Big/ \ln(\rho) \right)^{\frac{1}{\alpha}} - 1 \right\rceil. \tag{21}$$

4.5. Simulation study

Table 2, presents the maximum likelihood estimates of $DWG(p, \rho, \alpha)$ distribution and their standard errors for different values of n, of various simulated samples. Standard errors are attained by means of the asymptotic covariance matrix of the MLEs of parameters when the Newton–Raphson procedure converges.

4.6. Moments

The rth moment about origin is given by

$$\mu_r^{'} = E(Y^r) = \sum_{y=0}^{\infty} y^r \frac{(1-p)(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})}{(1 - p\rho^{y^\alpha})(1 - p\rho^{(y+1)^\alpha})}.$$

For the given values of p, ρ, and α, the moments can be numerically computed using R programming. Table 3 shows the moments, skewness, and kurtosis for DWG distribution for the given values of p, ρ, and α.

Table 3. Moments, skewness, and kurtosis for $p = 0.9,\ \rho = 0.9$ and various values of α.

Parameter	Raw moments	Central moments	Skewness	Kurtosis
$\alpha = 0.5$	$\mu_1' = 0.96$			
	$\mu_2' = 5.22$	$\mu_2 = 4.29$		
	$\mu_3' = 35.98$	$\mu_3 = 22.71$	6.53	9.11
	$\mu_4' = 279.68$	$\mu_4 = 167.61$		
$\alpha = 1.0$	$\mu_1' = 1.20$			
	$\mu_2' = 5.60$	$\mu_2 = 4.16$		
	$\mu_3' = 34.97$	$\mu_3 = 18.27$	4.63	7.62
	$\mu_4' = 257.59$	$\mu_4 = 257.59$		
$\alpha = 1.5$	$\mu_1' = 0.97$			
	$\mu_2' = 3.23$	$\mu_2 = 2.29$		
	$\mu_3' = 15.84$	$\mu_3 = 8.27$	5.69	10.01
	$\mu_4' = 98.35$	$\mu_4 = 52.47$		
$\alpha = 2.0$	$\mu_1' = 0.73$			
	$\mu_2' = 1.51$	$\mu_2 = 0.98$		
	$\mu_3' = 4.32$	$\mu_3 = 1.71$	3.06	7.80
	$\mu_4' = 15.90$	$\mu_4 = 15.90$		
$\alpha = 5$	$\mu_1' = 0.487$			
	$\mu_2' = 0.484$	$\mu_2 = 0.26$		
	$\mu_3' = 0.498$	$\mu_3 = 0.02$	0.03	1.25
	$\mu_4' = 0.527$	$\mu_4 = 0.08$		

4.7. Order statistics

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from $DWG(p, \rho, \alpha)$. Also, let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$, denote the corresponding order statistics. Then, the pmf and the cdf of kth order statistic, say, $Z = Y_{(k)}$, are given by

$$f_Z(z) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(z)\,[1-F(z)]^{n-k} f(z)$$
$$= \frac{n!}{(k-1)!(n-k)!} \frac{(1-p)^{(n-k+1)}\rho^{(n-k)(z+1)^\alpha}(1-\rho^{(z+1)^\alpha})^{k-1}(\rho^{z^\alpha}-\rho^{(z+1)^\alpha})}{(1-p\rho^{z^\alpha})(1-p\rho^{(z+1)^\alpha})^n}, \quad (22)$$

and

$$F_Z(z) = \sum_{j=k}^{n} \binom{n}{j} F^j(z)[1-F(z)]^{n-j}$$
$$= \sum_{j=k}^{n} \binom{n}{j} \frac{(1-p)^{n-j}\rho^{(n-j)(z+1)^\alpha}(1-\rho^{(z+1)^\alpha})^j}{(1-p\rho^{(z+1)^\alpha})^n}, \quad (23)$$

respectively.

The pmf of the minimum is

$$f_{Y_{(1)}}(z) = \frac{n(1-p)^n\rho^{(n-1)(z+1)^\alpha}(\rho^{z^\alpha}-\rho^{(z+1)^\alpha})}{(1-p\rho^{z^\alpha})(1-p\rho^{(z+1)^\alpha})^n}, \quad (24)$$

and the pmf of the maximum is

$$f_{Y_{(n)}}(z) = \frac{n(1-p)(1-\rho^{(z+1)^\alpha})^{n-1}(\rho^{z^\alpha}-\rho^{(z+1)^\alpha})}{(1-p\rho^{z^\alpha})(1-p\rho^{(z+1)^\alpha})^n}. \quad (25)$$

4.8. Maximum likelihood estimation (MLE)

Consider a random sample $(y_1, y_2, \ldots, y_n)$ of size n, from the $DWG(p, \rho, \alpha)$. Then, the log likelihood function is given by,

$$\log(L) = n\log(1-p) + \sum_{i=1}^{n}\log(\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha}) - \sum_{i=1}^{n}\log(1-p\rho^{y_i^\alpha}) - \sum_{i=1}^{n}\log(1-p\rho^{(y_i+1)^\alpha}). \tag{26}$$

The likelihood equations are,

$$\frac{\partial\log(L)}{\partial p} = \frac{-n}{1-p} + \sum_{i=1}^{n}\frac{\rho^{y_i^\alpha}}{1-p\rho^{y_i^\alpha}} + \sum_{i=1}^{n}\frac{\rho^{(y_i+1)^\alpha}}{1-p\rho^{(y_i+1)^\alpha}} = 0, \tag{27}$$

$$\frac{\partial\log(L)}{\partial \rho} = \sum_{i=1}^{n}\frac{y_i^\alpha\rho^{y_i^\alpha-1} - (y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}}{\rho^{y_i^\alpha}-\rho^{(y_i+1)^\alpha}} + p\sum_{i=1}^{n}\frac{y_i^\alpha\rho^{y_i^\alpha-1}}{1-p\rho^{y_i^\alpha}} + p\sum_{i=1}^{n}\frac{(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}}{1-p\rho^{(y_i+1)^\alpha}} = 0, \tag{28}$$

and

$$\frac{\partial\log(L)}{\partial \alpha} = \log(\rho)\sum_{i=1}^{n}\frac{y_i^\alpha\rho^{y_i^\alpha}\log(y_i) - (y_i+1)^\alpha\rho^{(y_i+1)^\alpha}\log(y_i+1)}{\rho^{y_i^\alpha}-\rho^{(y_i+1)^\alpha}} + p\log(\rho)\sum_{i=1}^{n}\frac{y_i^\alpha\rho^{y_i^\alpha}\log(y_i)}{1-p\rho^{y_i^\alpha}} + p\log(\rho)\sum_{i=1}^{n}\frac{(y_i+1)^\alpha\rho^{(y_i+1)^\alpha}\log(y_i+1)}{1-p\rho^{(y_i+1)^\alpha}} = 0. \tag{29}$$

These equations do not have explicit solutions and they have to be obtained numerically by using statistical softwares like *nlm* package in R programming. Let the estimators be $\hat{\theta} = (\hat{p}, \hat{\rho}, \hat{\alpha})^T$. The Fisher's information matrix is given by,

$$I_y(\theta) = \begin{bmatrix} -E\left(\frac{\partial^2 L}{\partial p^2}\right) & -E\left(\frac{\partial^2 L}{\partial p\partial\rho}\right) & -E\left(\frac{\partial^2 L}{\partial p\partial\alpha}\right) \\ -E\left(\frac{\partial^2 L}{\partial \rho\partial p}\right) & -E\left(\frac{\partial^2 L}{\partial \rho^2}\right) & -E\left(\frac{\partial^2 L}{\partial \rho\partial\alpha}\right) \\ -E\left(\frac{\partial^2 L}{\partial \alpha\partial p}\right) & -E\left(\frac{\partial^2 L}{\partial \alpha\partial\rho}\right) & -E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) \end{bmatrix}.$$

Here, the DWG family satisfies the regularity conditions which are fulfilled for the parameters in the interior of the parameter space, but not on the boundary (see Ferguson 1996). Hence, the vector $\hat{\theta}$ is consistent and asymptotically normal. That is, $\sqrt{I_y(\theta)}[\hat{\theta}-\theta]$ converges in distribution to multivariate normal with zero mean vector and identity covariance matrix.

The Fisher's information matrix can be computed using the approximation,

$$I_y(\hat{\theta}) \approx \begin{bmatrix} -\frac{\partial^2 L}{\partial p^2}\Big|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial p \partial \rho}\Big|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial p \partial \alpha}\Big|_{(\hat{p},\hat{\rho},\hat{\alpha})} \\ -\frac{\partial^2 L}{\partial \rho \partial p}\Big|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial \rho^2}\Big|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial \rho \partial \alpha}\Big|_{(\hat{p},\hat{\rho},\hat{\alpha})} \\ -\frac{\partial^2 L}{\partial \alpha \partial p}\Big|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial \alpha \partial \rho}\Big|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial \alpha^2}\Big|_{(\hat{p},\hat{\rho},\hat{\alpha})} \end{bmatrix},$$

where $\hat{p}$, $\hat{\rho}$, and $\hat{\alpha}$ are the MLEs of p, ρ, and α, respectively.

4.9. Stress–strength parameter

The stress–strength parameter, $R = P(Y > Z)$, is a measure of component reliability. Suppose that the random variable Y is the strength of a component which is subjected to a random stress Z. Estimation of R when Y and Z are iid has been considered in the literature. One may see Kotz, Lumelskii, and Pensky (2003), for a review of stress–strength model. In the discrete case, the stress–strength model is defined as

$$P(Y > Z) = \sum_{y=0}^{\infty} p_Y(y) F_Z(y),$$

where $p_Y(y)$ and $F_Z(y)$ denote the pmf and cdf of the independent discrete random variables Y and Z, respectively. Let $Y \sim DWG(\theta_1)$ and $Z \sim DWG(\theta_2)$, where $\theta_1 = (p_1,\ \rho_1,\ \alpha_1)^T$ and $\theta_2 = (p_2,\ \rho_2,\ \alpha_2)^T$. Then, from (10) and (12), we have,

$$R = P(Y > Z) = \sum_{y=0}^{\infty} \frac{(1-p_1)\left(\rho_1^{y^{\alpha_1}} - \rho_1^{(y+1)^{\alpha_1}}\right)\left(1-\rho_2^{(y+1)^{\alpha_2}}\right)}{\left(1-p_1\rho_1^{y^{\alpha_1}}\right)\left(1-p_1\rho_1^{(y+1)^{\alpha_1}}\right)\left(1-p_2\rho_2^{(y+1)^{\alpha_2}}\right)}. \tag{30}$$

Assume that $(y_1, y_2, \ldots, y_n)$ and $(z_1, z_2, \ldots, z_m)$ are independent observations drawn from $DWG(\theta_1)$ and $DWG(\theta_2)$, respectively. The total likelihood function is given by $L_R(\theta^*) = L_n(\theta_1)\, L_m(\theta_2)$, where $\theta^* = (\theta_1,\ \theta_2)$. The score vector is given by,

$$U_R(\theta^*) = \left(\frac{\partial L_R}{\partial p_1}, \frac{\partial L_R}{\partial \rho_1}, \frac{\partial L_R}{\partial \alpha_1}, \frac{\partial L_R}{\partial p_2}, \frac{\partial L_R}{\partial \rho_2}, \frac{\partial L_R}{\partial \alpha_2}\right).$$

The MLE $\hat{\theta^*}$ may be obtained from the solution of the non linear equation, $U_R(\hat{\theta^*}) = 0$. Applying $\hat{\theta^*}$ in Equation (30), the stress–strength parameter R can be obtained.

5. Applications

In this section, to show how the $DWG(p, \rho, \alpha)$ distribution works in practice, we use two real datasets, of which the first dataset is a discrete version of a continuous data and the second dataset is a count data. The parameters are estimated by using the mehod of maximum likelihood. We compare the fit of the DWG distribution with the following discrete lifetime distributions:

(a) Geometric (G) distribution having pmf,

$$P(Y = y) = (1-p)p^y; \quad 0 < p < 1, y = 0, 1, 2, \ldots.$$

(b) DW distribution having pmf,

$$P(Y = y) = q^{y^\beta} - q^{(y+1)^\beta}; \quad 0 < q < 1, \beta > 0, y = 0, 1, 2, \ldots.$$

(c) Discrete logistic (DLOG) distribution (see Chakraborty and Chakravarty 2016) having pmf

$$P(Y = y) = \frac{(1-p)p^{y-\mu}}{(1+p^{y-\mu})(1+p^{(y-\mu+1)})}; \quad 0 < p < 1, -\infty < \mu < \infty, \quad y \epsilon Z$$

(d) Exponentiated discrete Weibull (EDW) distribution (see Nekoukhou and Bidram 2015) having pmf,

$$P(Y = y) = (1 - p^{(y+1)^{\alpha}})^{\gamma} - (1 - p^{y^{\alpha}})^{\gamma}; \quad 0 < p < 1, \quad \alpha > 0, \quad \gamma > 0, \; y = 0, 1, 2, \ldots .$$

The values of the log-likelihood function ($-\log L$), the statistics K-S (Kolmogorov–Smirnov), AIC (Akaike information criterion), $AICC$ (Akaike information criterion with correction), and BIC (Bayesian information criterion) are calculated for the five distributions in order to verify which distribution fits better to these data. The better distribution corresponds to smaller $K-S$, $-\log L$, AIC, $AICC$, and BIC values. Here, $AIC = -2\log L + 2k$, $AICC = -2\log L + (\frac{2kn}{n-k-1})$ and $BIC = -2\log L + k\log n$ where L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size. The K-S distance, $D_n = \sup_y |F(y) - F_n(y)|$, where $F_n(y)$ is the empirical distribution function.

5.1. First dataset

The first dataset represents remission times (in months) of 128 bladder cancer patients (Lee and Wang 2003). The data are :

0.080 0.200 0.400 0.500 0.510 0.810 0.900 1.050 1.190 1.260 1.350 1.400 1.460 1.760 2.020 2.020 2.070 2.090 2.230 2.260 2.460 2.540 2.620 2.640 2.690 2.690 2.750 2.830 2.870 3.020 3.250 3.310 3.360 3.360 3.480 3.520 3.570 3.640 3.700 3.820 3.880 4.180 4.230 4.260 4.330 4.340 4.400 4.500 4.510 4.870 4.980 5.060 5.090 5.170 5.320 5.320 5.340 5.410 5.410 5.490 5.620 5.710 5.850 6.250 6.540 6.760 6.930 6.940 6.970 7.090 7.260 7.280 7.320 7.390 7.590 7.620 7.630 7.660 7.870 7.930 8.260 8.370 8.530 8.650 8.660 9.020 9.220 9.470 9.740 10.06 10.34 10.66 10.75 11.25 11.64 11.79 11.98 12.02 12.03 12.07 12.63 13.11 13.29 13.80 14.24 14.76 14.77 14.83 15.96 16.62 17.12 17.14 17.36 18.10 19.13 20.28 21.73 22.69 23.63 25.74 25.82 26.31 32.15 34.26 36.66 43.01 46.12 79.05.

Since the dataset is continuous, here first we discretize the data by considering the floor value (y).

The values in Table 4 indicate that the DWG distribution leads to a better fit compared to the other four models. Figure 3 shows that the fitted pmf for DWG model is closer to the empirical histogram than the other four models considered.

Table 4. Parameter estimates and goodness of fit for various models fitted for the first dataset.

Model	MLEs	−log L	AIC	AICC	BIC	K-S	p value
G	$\hat{p} = 0.8991$	414.836	831.672	831.704	831.779	0.1000	0.1549
DW	$\hat{q} = 0.9114$	414.556	833.112	837.304	833.326	0.1131	0.0758
	$\hat{\beta} = 1.0511$						
DLOG	$\hat{p} = 0.8000$	456.825	917.650	917.746	917.412	0.1860	0.0003
	$\hat{\mu} = 7.6149$						
EDW	$\hat{p} = 0.4689$	409.766	825.532	825.726	825.174	0.1237	0.0399
	$\hat{\alpha} = 0.5397$						
	$\hat{\gamma} = 4.9697$						
DWG	$\hat{p} = 0.9529$	409.277	824.554	824.748	824.196	0.0905	0.2458
	$\hat{\rho} = 0.9982$						
	$\hat{\alpha} = 1.7025$						

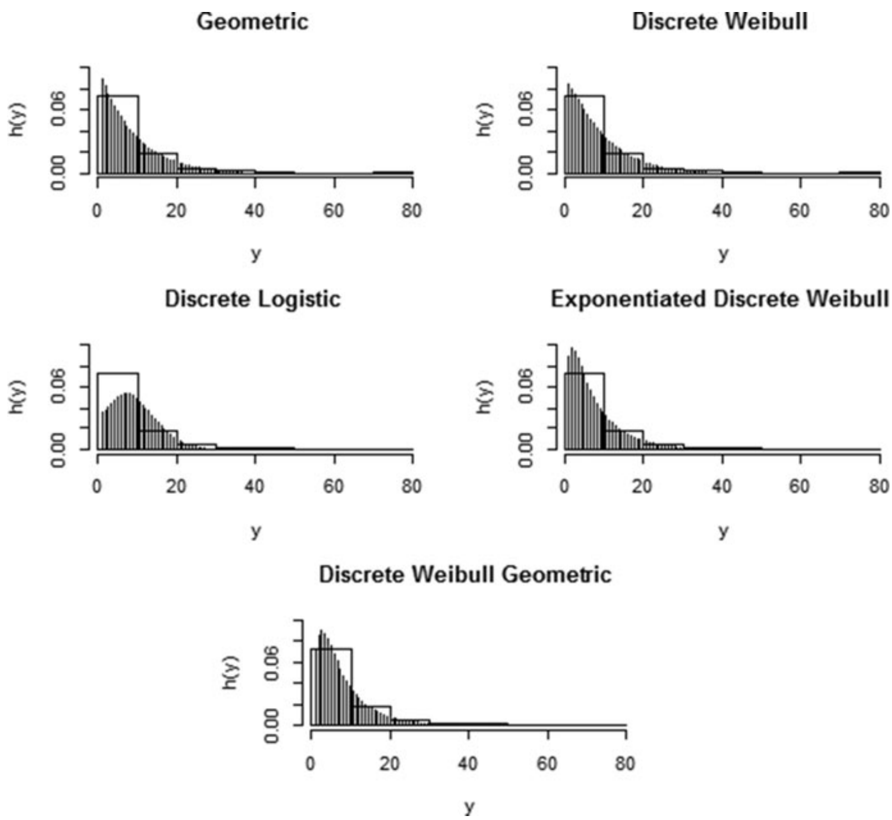

Figure 3. Fitted models for the first dataset.

Figure 4 shows the structure of the cdf of the five models with the empirical distribution of the given data. Here, the dotted line indicates the empirical cdf of the data.

5.2. Second dataset

The second dataset is the number of shocks before failure reported in Murthy, Xie, and Jiang (2004, p. 245). The data are:

$$1,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 6,\ 6,\ 7,\ 10,\ 11,\ 12,\ 14.$$

Table 5. Parameter estimates and goodness of fit for various models fitted for the second dataset.

Model	MLEs	−log L	AIC	AICC	BIC	*K-S*	*p* value
G	$\hat{p} = 0.8609$	46.389	94.778	95.064	93.982	0.2995	0.1133
DW	$\hat{q} = 0.9831$	41.637	89.274	90.197	87.682	0.2227	0.4057
	$\hat{\beta} = 2.0111$						
DLOG	$\hat{p} = 0.6079$	43.170	90.340	91.263	88.748	0.1851	0.6435
	$\hat{\mu} = 6.2330$						
EDW	$\hat{p} = 0.7183$	41.224	88.454	90.454	86.066	0.1868	0.6317
	$\hat{\alpha} = 1.0020$						
	$\hat{\gamma} = 4.6559$						
DWG	$\hat{p} = 0.8637$	41.050	88.100	90.100	85.712	0.1715	0.7341
	$\hat{\rho} = 0.9993$						
	$\hat{\alpha} = 2.8921$						

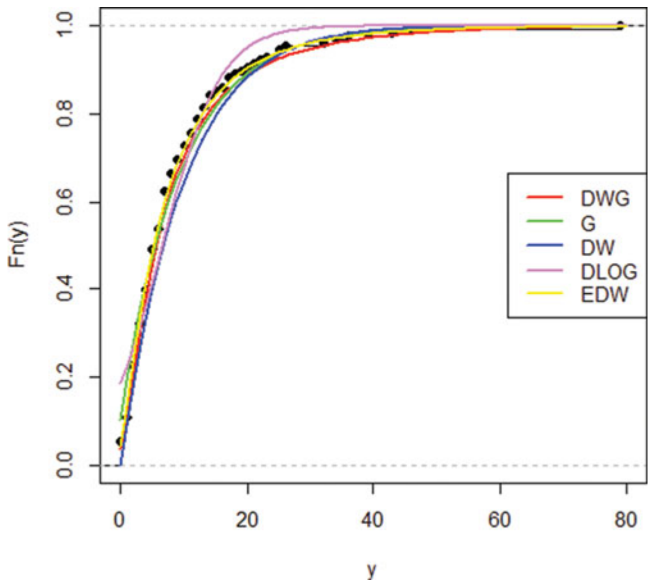

Figure 4. Empirical and fitted cdfs for the first dataset.

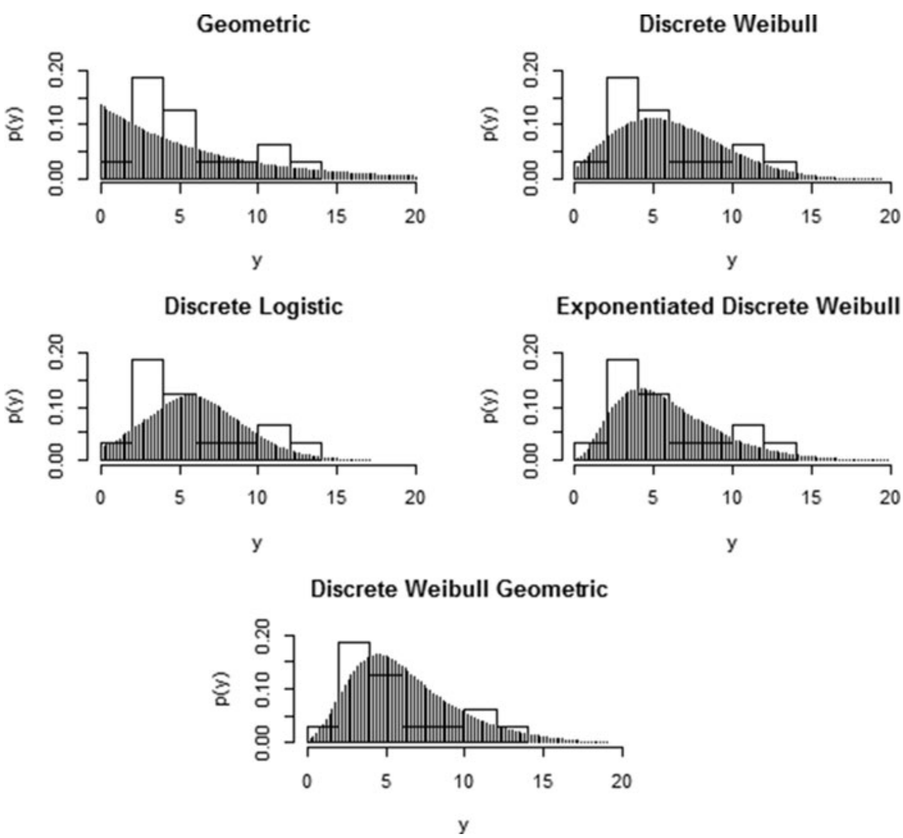

Figure 5. Fitted models for the second dataset.

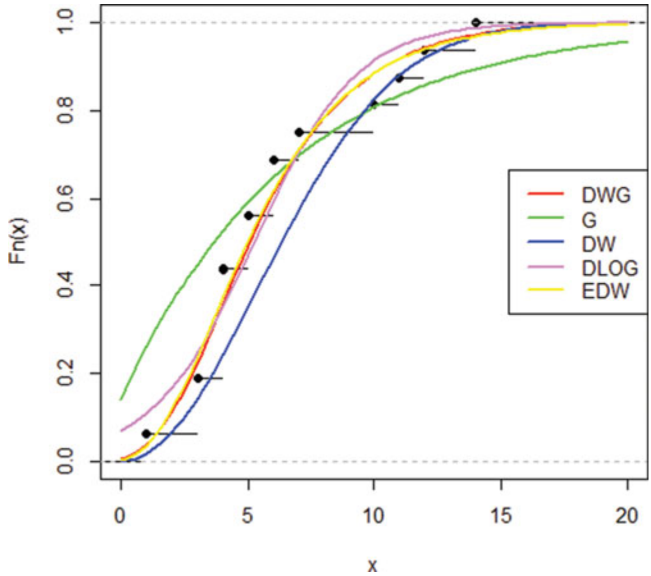

Figure 6. Empirical and fitted cdfs for the second dataset.

The values in Table 5 indicate that the *DWG* distribution leads to a better fit for the second dataset compared to the other four models.

Figure 5 shows that the fitted pmf for the *DWG* model is closer to the empirical histogram than the other four models considered.

Figure 6 shows the structure of the cdf of the five models in comparison with the empirical distribution function of the given data. The dotted line indicates the empirical cdf of the second dataset.

6. Conclusion

In the present study, we introduce the discrete Weibull geometric (DWG) distribution. This discrete distribution contains the DW, discrete Rayleigh, and geometric distribution as special cases . We have studied the basic statistical and mathematical properties of the new model and illustrated that the hazard rate function of the new model can be increasing or decreasing. Fitting the DWG model with two real datasets indicates the flexibility and capacity of the new distribution in data modeling.

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